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Semiparametric estimation and model selection for conditional mixture copula models

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Abstract

Conditional copula models allow the dependence structure among variables to vary with covariates, and thus can describe the evolution of the dependence structure with those factors. This paper proposes a conditional mixture copula which is a weighted average of several individual conditional copulas. We allow both the weights and copula parameters to vary with a covariate so that the conditional mixture copula offers additional flexibility and accuracy in describing the dependence structure. We propose a two-step semi-parametric estimation method and develop asymptotic properties of the estimators. Moreover, we introduce model selection procedures to select the component copulas of the conditional mixture copula model. Simulation results suggest that the proposed procedures have a good performance in estimating and selecting conditional mixture copulas with different model specifications. The proposed model is then applied to investigate how the dependence structures among international equity markets evolve with the volatility in the exchange rate markets.

KEYWORDS

conditional copula, mixture copula, model selection, semi-parametric estimation

1 INTRODUCTION

The Sklar's theorem by Sklar (1959) enables one to decompose a multivariate joint density into a product of univariate marginal densities and a copula density, so that the latter contains all information about the dependence structure. A copula model has several desirable properties when applied to study dependence. For example, a copula model can catch various types of dependence structures such as linear or nonlinear, symmetric, or asymmetric, tail or nontail dependence. Moreover, unlike the conventional linear correlation, it is invariant to strictly monotonic transformations. Patton (2012) and Fan and Patton (2014) provide excellent summaries of the development in copula.

In many cases, researchers may need a conditional copula to better describe the effect of a covariate on the degree of dependence. In a conditional copula model, the degree of dependence, measured by the copula parameter, is no longer a constant but a function of a covariate. Therefore, compared with copulas with constant parameters, a conditional copula provides another channel to investigate the dependence structure among variables. In the literature, Patton (2006) pioneers the conditional copula model by extending the Sklar's theorem for conditional distributions and sets the copula parameter to be a parametric function of lagged terms. After that, there is a sequence of studies concentrating on the dynamic in copula parameter (e.g., Acar et al., 2011; Abegaz et al., 2012; Fermanian & Lopez, 2018; Giacomini et al., 2009; Garcia & Tsafack, 2011; Hafner & Manner, 2012).

Another line of extension is to propose a mixture copula that is a linear combination of several individual copulas. The key idea is that, by combining individual copulas with different dependence patterns, a mixture copula can capture dependence structures which do not belong to any individual copula, and thus exhibits greater flexibility to describe dependence structures. Therefore, a mixture copula is more flexible than an individual copula and can be used to specify various dependence structures in data (e.g., Cai & Wang, 2014; Chollete et al., 2005; Hu, 2006; Liu et al., 2019). However, even though a mixture copula exhibits great flexibility in describing more general dependence structures, the parameters in this model—weights and copula parameters—are usually assumed to be constants, so researchers still face difficulty in describing how the dependence evolves with certain covariates.

In this paper, we contribute to the literature by proposing an innovative semi-parametric conditional mixture copula model which allows both the weights and copula parameters in the mixture copula model to vary with a covariate in a nonparametric way. The superiority of a conditional mixture copula is that it carries the advantages of both the conditional copula and mixture copula discussed above, so empirical practitioners can flexibly describe the dependence structure and effectively mitigate the potential model misspecification problem simultaneously. For example, compared with an individual conditional copula, a conditional mixture copula offers additional flexibility and accuracy by accommodating more copula families whose weights and parameters are both varying with a covariate. To estimate the unknown parameters, we maximize a local log-likelihood function by applying the local polynomial framework (Fan & Gijbels, 1996). We then establish the large sample properties of the nonparametric estimators under some regularity conditions.

When investigating the choice of an appropriate conditional mixture copula model, we suggest two copula model selection methods. The first one follows Huang et al. (2013) who propose an information criterion approach to select the components in the nonparametric mixture of regression models. Specifically, for each candidate mixture model consisting of different combinations of individual copulas, we calculate its maximum log-likelihood and then construct an information criterion such as BIC. To implement such an information criterion, we will need an extra step to consider the model complexity, which is measured by the degree of freedom derived by Fan et al. (2001). Then, the model with the lowest information criterion value will be selected from the candidate models. The second model selection strategy is in spirit similar to the backward elimination or forward addition procedures in the context of linear regression settings. It involves a sequence of generalized likelihood ratio tests through which component copulas with insignificant weights at the conventional significance levels are filtered out. Using either method, we achieve the goal of selecting an appropriate conditional mixture copula from all candidates to best describe the dependence structure, and then estimate the unknown parameters in the selected model.

Our simulation results show that the proposed estimation method and model selection procedures exhibit a good performance when the true model is either an individual copula or a mixture copula. On one hand, the estimation errors of copula parameters and weights associated with each component copula decrease remarkably when the sample size increases. On the other hand, the true component copulas are highly likely to be selected even when the sample size is small, and the probability of inaccurate selection declines as the sample size increases.

In an empirical illustration, we apply the proposed estimation and model selection procedures to investigate the dependence structures and comovement patterns among the equity returns in France, Germany, the United States, and the United Kingdom along the volatility in the exchange rate markets of the four countries. The empirical results show that, of the Clayton, Gumbel and Frank copulas, the Clayton and Frank copulas are always selected and the weight of the Clayton copula increases when the exchange rates become extremely volatile, indicating a more salient lower tail dependence among the equity markets in the four economies. When examining the magnitude of the dependence measured by Kendall's τ, we find that the lower tail dependence becomes strengthened as volatility in the exchange rate markets increases. Both findings are in line with Garcia and Tsafack (2011): when a sudden shock hits an economy with an active currency market, transmission through the exchange rate market leads to a downside comovement of equity markets more likely than in a tranquil period of the exchange rate market.

The rest of the paper is organized as follows. In Section 2, we propose the estimation method, the asymptotic theory, and the model selection procedures for conditional mixture copula models. We conduct Monte Carlo simulations and discuss the results in Section 3. To highlight the practical usefulness of the proposed methods, in Section 4 we provide an empirical illustration on how the dependence structures among international equity markets evolve with the volatility in exchange rate markets. Section 5 draws the conclusion. In Data S1, Appendix A provides a stationary bootstrap technique, Appendix B discusses some practical issues including an EM algorithm, the selection of the bandwidth, and the confidence intervals, Appendix C documents the proofs of the key results, and Appendix D presents additional simulation results.

2 | MODEL AND ESTIMATION

In this section, we present a semi-parametric conditional mixture copula model and the corresponding estimation and selection procedures.

2.1 A semi-parametric conditional mixture copula model

Let $\{\mathbf{X}_t\}_{t=1}^T$ be a series of p-dimensional vectors with $\mathbf{X}_t = \left(X_{1t}, \dots, X_{pt}\right)^\top$ and p being a finite positive integer, and let $\{Z_t\}_{t=1}^T$ be a 1-dimensional vector of the covariate. Denote $F(\mathbf{x}_t|z_t)$ and $f(\mathbf{x}_t|z_t)$ as the joint distribution and the density function of $\mathbf{X}_t$ evaluated at $\mathbf{x}_t \in \mathcal{R}^p$ and conditional on $Z_t = z_t$, $F_s(x_{st}|z_t)$ and $f_s(x_{st}|z_t)$ as the marginal distribution and the density function of X_{st} evaluated at $x_{st} \in \mathcal{R}$ and conditional on $Z_t = z_t$, respectively, where $s = 1, \dots, p$. Our target is to estimate the conditional joint distribution $F(\mathbf{x}_t|z_t)$ based on a conditional mixture copula model. Theoretically, the conditional mixture copula model can be formulated as a linear combination of infinite individual copulas:

$$C\{u(z_t); \omega(z_t), \theta(z_t)\} = \sum_{k=1}^{\infty} \omega_k(z_t) C_k\{u(z_t); \theta_k(z_t)\},$$

where $\{C_k(\cdot;\cdot)\}_{k=1}^{\infty}$ is a set of candidate copulas with unknown parameters $\{\theta_k\}$ and a p-dimensional conditional marginal distribution $u(z_t) = (F_1(x_{1t}|z_t), \dots, F_p(x_{pt}|z_t))$. $\{C_k(\cdot;\cdot)\}_{k=1}^{\infty}$ can be regarded as known basis copula functions so that $C\{u(z_t); \omega(z_t), \theta(z_t)\}$ can be regarded as a series expansion based on the basis copula functions $\{C_k(\cdot;\cdot)\}_{k=1}^{\infty}$. In real applications, we use finite number of d individual copulas to approximate the true model:

$$C\{u(z_t); \omega(z_t), \theta(z_t)\} = \sum_{k=1}^{d} \omega_k(z_t) C_k\{u(z_t); \theta_k(z_t)\}, \tag{1}$$

where $\omega(z_t) = (\omega_1(z_t), \dots, \omega_d(z_t))^\top$, $\theta(z_t) = (\theta_1(z_t), \dots, \theta_d(z_t))^\top$, and $\{C_1(\cdot\,;\cdot), \dots, C_d(\cdot\,;\cdot)\}$ is a set of candidate copulas. Let $\{\omega_k\}_{k=1}^d$ denote the weight parameters satisfying $0 \le \omega_k \le 1$ and $\sum_{k=1}^d \omega_k = 1$, and d is the number of candidate copulas. The copula parameters $\{\theta_k(z_t)\}$ and the weight parameters $\{\omega_k(z_t)\}$ are set to be unknown functions of the covariate.

When using (1) to approximate the true model, we may encounter a misspecification problem because some true individual copulas might not be included. To avoid this problem, we can first consider a large set of candidate copulas and then employ a copula model selection procedure discussed in Section 2.4 to filter out the "insignificant" component copulas. Furthermore, even if some true individual copulas are excluded so that the model becomes misspecified, we can still estimate and select the closest mixture copula model by the model selection procedure described in Section 2.4. Therefore, the model in (1) is flexible enough to capture a true copula in real applications.

For model identification, two conditional mixture copulas $C\{u(z_t); \omega(z_t), \theta(z_t)\} = \sum_{k=1}^d \omega_k(z_t) C_k\{u(z_t); \theta_k(z_t)\}$ and $C^*\{u(z_t); \omega^*(z_t), \theta^*(z_t)\} = \sum_{k=1}^{d^*} \omega_k^*(z_t) C_k^*\{u(z_t); \theta_k^*(z_t)\}$ are said to be identified, that is, $C\{u(z_t); \omega(z_t), \theta(z_t)\} \equiv C^*\{u(z_t); \omega^*(z_t), \theta^*(z_t)\}$, if and only if $d = d^*$ and we can order the summations such that $\omega(z_t) = \omega^*(z_t)$ and $C_k\{u(z_t); \theta_k(z_t)\} = C_k^*\{u(z_t); \theta_k^*(z_t)\}$ for all possible values of z_t, $u(z_t)$ with $k = 1, \dots, d$. Without loss of generality, we follow Cai and Wang (2014) and assume that the conditional mixture copula model under investigation is identified.

Our model setting here has three superiorities. First, instead of imposing assumptions on the functional forms of the unknown weights and copula parameters, we conduct a data-driven method (which will be specified below) to estimate them. Second, compared with an individual conditional copula, our conditional mixture copula model allows not only the copula parameters but also the weight parameters of the component copulas to vary with the covariate. Third, when constructing the model in Equation (1), we impose no restrictions on the number of candidate copulas included in the model so that a large copula candidate set can be taken to avoid the copula misspecification problem. We will filter out the “insignificant” component copulas through the copula selection procedures discussed later in Section 2.4.

2.2 | Estimation procedures

We propose to estimate model (1) in two steps. First, we estimate the unknown marginal distributions in the model by a rescaled empirical distribution function method. Second, after replacing the unknown marginal distributions with the estimates obtained from the first step, we adopt the local polynomial approximation (see Fan & Gijbels, 1996) in a local log-likelihood setting to estimate the weights and copula parameters in model (1). Each step is described specifically as follows:

Step One: We follow Chen and Fan (2006a) and use a rescaled empirical distribution function to estimate the marginal distributions, that is,

$$\hat{F}_s(x_{st}) = \frac{1}{T+1}\sum_{t=1}^{T} I\{X_{st} \le x_{st}\} \quad \text{for} \quad s = 1, \dots, p.$$

Remark 1. Ideally, one should use conditional estimators to estimate the marginal distributions, that is, $\hat{F}_s(x_{st}|z_t) := \sum_{t=1}^{T} I\{X_{st} \le x_{st}\}K_h(Z_t - z_t)/\sum_{t=1}^{T} K_h(Z_t - z_t)$ (see Abegaz et al., 2012). However, due to the fact that estimators for both the marginals and copula parameters have the same convergence rate of $\sqrt{Th}$, this setting would make the asymptotic properties of copula parameters more complicated, especially with time series data. Because the main focus of this paper is on the estimation of weights and copula parameters of a conditional mixture copula model and selection of component copula families, we assume that the marginal distributions do not depend on the covariate, which is similar to Acar et al. (2011). Relaxing this assumption for marginal distributions would be an interesting topic for future research.

Step Two: Given the estimators of the marginals $\hat{u}_t = \left(\hat{F}_1(x_{1t}), \dots, \hat{F}_p(x_{pt})\right)^\top = \left(\hat{u}_{1t}, \dots, \hat{u}_{pt}\right)^\top$, we next estimate the unknown weight and copula parameters locally by a polynomial. The copula parameter space is restricted for many widely used copula families. For example, for a Gaussian copula, $\theta \in (-1, 1)$, and for a Gumbel Copula, $\theta \in [1, \infty)$. Moreover, the weight parameters are restricted to take values between 0 and 1. In contrast, the polynomial framework assumes that any points belong to $\mathcal{R}$ can be taken. For this reason, we follow Acar et al. (2011) and Abegaz et al. (2012) and use some known inverse transformation functions to ensure that the weight and copula parameter space is correct. Specifically, we denote $g_{\omega,k}^{-1} : \mathcal{R} \to \Omega_k$ and $g_{\theta,k}^{-1} : \mathcal{R} \to \Theta_k$ as the inverse link functions respectively for the weight and copula parameters of the kth component copula. Therefore, we have $\omega_k(z) = g_{\omega,k}^{-1}(w_k(z)) \in \Omega_k$ and $\theta_k(z) = g_{\theta,k}^{-1}(\vartheta_k(z)) \in \Theta_k$ for $k = 1, \dots, d$. The choice of the link functions is not important for theory development as long as they are monotone. For example, we can choose the inverse link functions $g^{-1}(z) = \exp(z)$ for the Clayton copula, $g^{-1}(z) = z$ for the Frank copula, and $g^{-1}(z) = \exp(z) + 1$ for the Gumbel copula, so that the resulting copula parameter estimates are guaranteed to be in the correct range.

Then, for $k = 1, \dots, d$, assuming that w_k and ϑ_k have the $(q+1)$th derivative at point z, we can approximate $w_k(z_t)$ and $\vartheta_k(z_t)$ for data points z_t in the neighborhood of z by the following Taylor expansions:

$$\begin{aligned} w_k(z_t) &\approx w_k(z) + w_k^{(1)}(z)(z_t - z) + \dots + w_k^{(q)}(z)(z_t - z)^q/q! \\ &\equiv \alpha_{k0} + \alpha_{k1}(z_t - z) + \dots + \alpha_{kq}(z_t - z)^q, \end{aligned} \tag{2}$$

$$\begin{aligned} \vartheta_k(z_t) &\approx \vartheta_k(z) + \vartheta_k^{(1)}(z)(z_t - z) + \dots + \vartheta_k^{(q)}(z)(z_t - z)^q/q! \\ &\equiv \beta_{k0} + \beta_{k1}(z_t - z) + \dots + \beta_{kq}(z_t - z)^q, \end{aligned} \tag{3}$$

where $\alpha_{kr} = \alpha_{kr}(z) = w_k^{(r)}(z)/r!$ and $\beta_{kr} = \beta_{kr}(z) = \vartheta_k^{(r)}(z)/r!$ for each $r \in \{0, \dots, q\}$. Then, the local log-likelihood function can be approximated as follows:

$$\frac{1}{T}\sum_{t=1}^{T} \ln\left(\sum_{k=1}^{d} g_{\omega,k}^{-1}\left\{\alpha_{k0} + \dots + \alpha_{kq}(z_t - z)^q\right\} c_k\left[\hat{u}_t; g_{\theta,k}^{-1}\left\{\beta_{k0} + \dots + \beta_{kq}(z_t - z)^q\right\}\right]\right) \times K_h(z_t - z), \tag{4}$$

where $c_k(\cdot)$ is the copula density function of the kth component copula in model (1), and $K(\cdot)$ is a kernel function with $K_h(\cdot) = K(\cdot/h)/h$ and h being the bandwidth.

For the choice of q, we take $q = 1$ throughout this paper. That is, we apply the commonly used local linear fitting (see Fan & Gijbels, 1996) in the paper. The local log-likelihood function then reduces to

$$L(\hat{u}, \delta) = \frac{1}{T}\sum_{t=1}^{T} \ln\left(\sum_{k=1}^{d} g_{\omega,k}^{-1}\left\{\alpha_{k0} + \alpha_{k1}(z_t - z)\right\} c_k\left[\hat{u}_t; g_{\theta,k}^{-1}\left\{\beta_{k0} + \beta_{k1}(z_t - z)\right\}\right]\right) \times K_h(z_t - z), \tag{5}$$

where $\hat{u} = \left(\hat{u}_1^{\intercal}, \dots, \hat{u}_T^{\intercal}\right)^{\intercal}$ and $\delta = (\alpha_{10}, \dots, \alpha_{d0}, \beta_{10}, \dots, \beta_{d0}, \alpha_{11}, \dots, \alpha_{d1}, \beta_{11}, \dots, \beta_{d1})^{\intercal}$. Note that a maximum likelihood estimator may not have a closed form, so an iterative algorithm should be adopted to find the numerical solution (see Appendix B for details). Then we can obtain the estimators of $w_k^{(r)}(z)$ and $\vartheta_k^{(r)}(z)$ by defining $\hat{w}_k^{(r)}(z) = r!\hat{\alpha}_{kr}$ and $\hat{\vartheta}_k^{(r)}(z) = r!\hat{\beta}_{kr}$ with $r \in \{0, 1\}$. Finally, for any covariate z, the weight and copula parameters in model (1) can be respectively estimated by

$$\begin{aligned} \hat{\omega}_k(z) &= g_{\omega,k}^{-1}(\hat{w}_k(z)) = g_{\omega,k}^{-1}(\hat{\alpha}_{k0}), \quad \text{and} \\ \hat{\theta}_k(z) &= g_{\theta,k}^{-1}(\hat{\vartheta}_k(z)) = g_{\theta,k}^{-1}(\hat{\beta}_{k0}), \end{aligned}$$

for $k = 1, \dots, d$.

2.3 | Large sample theory

To find the large sample properties of the nonparametric estimators, first, we rewrite the kernel-based local log-likelihood function as

$$L(u, \delta) = \frac{1}{T}\sum_{t=1}^{T} \ell\left(u_t, g^{-1}(\tilde{Z}_t^{\intercal}\delta)\right) K_h(z_t - z),$$

where $\ell(u_t, g^{-1}(\tilde{Z}_t^\top \delta)) = \ln\left(\sum_{k=1}^{d} g_{\omega,k}^{-1}\{\alpha_{k0} + \alpha_{k1}(z_t - z)\}\, c_k[u_t; g_{\theta,k}^{-1}\{\beta_{k0} + \beta_{k1}(z_t - z)\}]\right)$, $g^{-1}(\cdot) = \left(g_{\omega,1}^{-1}(\cdot), \dots, g_{\omega,d}^{-1}(\cdot), g_{\theta,1}^{-1}(\cdot), \dots, g_{\theta,d}^{-1}(\cdot)\right)^\top$ is a vector of link functions, and $\tilde{Z}_t = (I, (z_t - z)I)^\top$ with I being a $2d \times 2d$ identity matrix. Next, we define a vector of coefficients $\xi = (\omega^\top, \theta^\top)^\top = \left(g_{\omega,1}^{-1}(w_1), \dots, g_{\omega,d}^{-1}(w_d), g_{\theta,1}^{-1}(\vartheta_1), \dots, g_{\theta,d}^{-1}(\vartheta_d)\right)^\top$ and its corresponding estimator $\hat{\xi}$. Similarly, we define $\eta = (w^\top, \vartheta^\top)^\top = (w_1, \dots, w_d, \vartheta_1, \dots, \vartheta_d)^\top$ and its corresponding estimator $\hat{\eta}$. In addition, let $f(z)$ be the density function of z and ϵ be a small positive constant. Define the domain of z as $\Phi_z = \{z : f(z) \geq \epsilon$; there exists a and b such that $z \in [a, b]\}$, that is, Φ_z is the set of bounded z whose density is bounded away from 0.

Meanwhile, we introduce some regularity conditions as below:

- C1. The vector of functions η is continuous, bounded and has third order continuous derivatives on Φ_z;
- C2. There exists two constants a and b such that for any $z \in [a, b]$, the density function $f(z)$ is continuous and $f(z) > \epsilon$ for a small positive constant $\epsilon > 0$;
- C3. The copula log-likelihood function $\ell(u_t, \xi)$ has bounded third derivative with respect to ξ and bounded second derivative with respect to u_t. Further, $\partial \ell(u_t, \xi)/\partial \xi$ and $[g^{-1}]'$ are Lipschitz continuous;
- C4. $0 \leq \omega_k(z) \leq 1$ and $\sum_{k=1}^{d} \omega_k(z) = 1$ for all $z \in \Phi_z$;
- C5. The kernel function $K(z)$ is twice continuously differentiable on the support $(-1, 1)$, and its second order derivative satisfies a Lipschitz condition. Let $\nu_0 = \int K^2(z)dz$, $\nu_2 = \int z^2 K^2(z)dz$ and $\mu_2 = \int z^2 K(z)dz$;
- C6. The bandwidth h satisfies that $h \to 0$ and $Th \to \infty$, as $T \to \infty$;
- C7. Assume that $\{\mathbf{X}_t, Z_t\}_{t=1}^{T}$ is a strictly stationary α-mixing sequence. Furthermore, assume that there exists some constant $c > 0$ such that $E\|\mathbf{X}_t\|^{2(2+c)} < \infty$ where $\|\cdot\|$ represents the Euclidean norm (L_2-norm), $E|Z_t|^{2(2+c)} < \infty$, and the mixing coefficient $\alpha(m)$ satisfies $\alpha(m) = O(m^{-c_0})$ with $c_0 = (2+c)(1+c)/c$.

Remark 2. Conditions in C1–C3 are for the derivation of the asymptotic properties. Conditions in C4 are mild conditions for identification and conditions in C5 and C6 are commonly employed in nonparametric estimation. Conditions in C7 are the common conditions with weakly dependent data. Most financial models such as ARMA, ARCH, and GARCH models satisfy these conditions; see Cai (2002).

Theorem 1. *Let $\{\mathbf{X}_t, Z_t\}_{t=1}^{T}$ be a strictly stationary and strong mixing sequence. Assume that $\sup_{1\leq t\leq T}|\hat{u}_{st} - u_{st}| = O_p(1/\sqrt{T})$ for $s = 1, \dots, p$, $h \to 0$ and $Th \to \infty$ as $T \to \infty$. For a fixed point $z \in \Phi_z$, under conditions C1 - C7, we have*

$$D_T\left(\hat{\delta} - \delta - h^2 B(z)\right) \xrightarrow{d} N\left(0, \begin{pmatrix} \frac{\nu_0}{f(z)} \\ \frac{\nu_2}{\mu_2^2 f(z)} \end{pmatrix} \otimes \{\Psi(z) \circ \{[(g^{-1})'(\eta(z))][(g^{-1})'(\eta(z))]^\top\}^{-1}\}\right),$$

where $D_T = \mathrm{diag}(\sqrt{Th}I, \sqrt{Th}hI)$ with I being an $2d \times 2d$ identity matrix, $B(z) = \left(\frac{1}{2}\eta''(z)^\top \mu_2, 0^\top\right)^\top$ is the bias term, $\otimes$ is the Kronecker product, and $\circ$ is the Hadamard product. $\Psi(z) = \Sigma^{-1}(z)\Omega(z)\Sigma^{-1}(z)$ with $\Sigma(z) = -E\{\ell''(u_t, g^{-1}(\eta(z_t)))|z_t = z\}$ and $\Omega(z) = E\{\ell'(u_t, g^{-1}(\eta(z_t)))\ell'\left(u_t, g^{-1}(\eta(z_t))\right)^\top |z_t = z\}$.

Remark 3. The condition $\sup_{1\leq t\leq T}|\hat{u}_{st} - u_{st}| = O_p(1/\sqrt{T})$ for $s = 1, \dots, p$ can be obtained from lemma 4.1 in Chen and Fan (2006a). From Theorem 1, as expected, the marginal estimator $\hat{u}_t$ has little effect on $\hat{\delta}$ in a large sample, due to the fact that $\hat{u}_t$ is estimated at a faster convergence rate than the nonparametric estimator $\hat{\delta}$.

Corollary 1. *It follows from Theorem 1 that, for a fixed point $z \in \Phi_z$, as $T \to \infty$, we have*

$$\sqrt{Th}\left(\hat{\eta}(z) - \eta(z) - \frac{h^2}{2}\eta''(z)\mu_2\right) \xrightarrow{d} N\left(0, \frac{\nu_0}{f(z)}\Psi(z)\circ\{[(g^{-1})'(\eta(z))][(g^{-1})'(\eta(z))]^{\intercal}\}^{-1}\right).$$

Corollary 2. *By the continuity of the inverse link function $g^{-1}(\cdot)$, for a fixed point $z \in \Phi_z$, as $T \to \infty$, we have*

$$\sqrt{Th}\left(\hat{\xi}(z) - \xi(z) - h^2 B_{\xi}(z)\right) \xrightarrow{d} N\left(0, \frac{\nu_0}{f(z)}\Psi(z)\right),$$

where $B_{\xi}(z) = \frac{1}{2}\mu_2 \frac{1}{g'(\xi(z))} \circ \eta''(z)$.

2.4 | Model selection for conditional mixture copula models

When a mixture copula model contains too many component copulas, there is a risk of overfitting and efficiency loss. To filter out component copulas with small weights and little contribution to the dependence structure, we consider two model selection procedures.

The first method is to apply the information criterion such as AIC or BIC. However, as argued in Section 1, they cannot be directly used to the proposed conditional mixture copula model which has varying coefficients. Huang et al. (2013) use the BIC-type selector to identify the number of components in the nonparametric mixture of regression models and find it performs well in numerical studies. This motivates us to select the components in the conditional mixture copula model through the BIC-type selector.

Let $|\Phi_z|$ be the length of the support of z and $K * K$ be the convolution of the kernel K. Define $e_k = K(0) - 0.5\int K^2(t)dt$ and $m_k = \int (K(t) - 0.5K * K(t))^2 dt$. Following Huang et al. (2013), we can calculate the value of the BIC by

$$-2L + \log(T) \times df,$$

where

$$L = \frac{1}{T}\sum_{t=1}^{T} \ln\left\{\sum_{k=1}^{d} \hat{\omega}_k(z_t) c_k\left[\hat{u}_t; \hat{\theta}_k(z_t)\right]\right\},$$

and

$$df = (2d-1) r_k e_k |\Phi_z|/h \quad \text{with} \quad r_k = e_k/m_k.$$

Both $\hat{u}_t$ and $\hat{\xi} = \left(\hat{\omega}_1, \dots, \hat{\omega}_d, \hat{\theta}_1, \dots, \hat{\theta}_d\right)^{\intercal}$ can be obtained by the estimation procedures discussed in Section 2.2. The degree of freedom, which is originally derived for the generalized likelihood ratio test (GLRT) by Fan et al. (2001), can be understood as follows. Suppose that we partition the range of z into $|\Phi_z|/h$ intervals with equispaced length h. Hence, the effective number of each parameter is approximately proportional to $|\Phi_z|/h$. $r_k e_k$ is an adjusting factor that accounts for overlapping intervals due to the local linear fitting. $2d-1$ is the number of nonparametric estimators in which we minus one to account for the constraint on the weight parameters.

An alternative method to decide which component copulas should be kept or filtered out is to apply a sequence of hypothesis tests. In the classical linear regression models, a sequence of F-tests with the backward elimination or forward addition procedures is used to select the important regressors. We adopt a similar strategy and implement the following test:

$$H_0 : \omega_{i_1}(z) = \dots = \omega_{i_l}(z) = 0 \quad \text{versus} \quad H_1 : \quad \text{not all} \;\; \omega_{i_s}(z) = 0,$$

for some $\{i_1, \dots, i_l\} \subset \{1, \dots, d\}$. The model selection is achieved by a sequence of testing procedures above. To simplify the presentation, we only consider the following test:

$$H_0 : \omega_1(z) = \dots = \omega_J(z) = 0 \quad \text{versus} \quad H_1 : \quad \text{not all} \;\; \omega_j(z) = 0,$$

and other cases can be implemented in the same manner. Using the local linear fitting with a kernel K and a bandwidth h, we can obtain $\tilde{\xi}(z_t)$ and $\hat{\xi}(z_t)$ under the null hypothesis H_0 and the alternative hypothesis H_1, respectively.

Define

$$L(H_0) = \sum_{t=1}^{T} \ell(\hat{u}_t, \tilde{\xi}(z_t)) \quad \text{and} \quad L(H_1) = \sum_{t=1}^{T} \ell(\hat{u}_t, \hat{\xi}(z_t)),$$

where $\hat{u}_t$ denotes the estimator of the marginal distribution. Fan et al. (2001) propose a GLRT statistic that can be used in many nonparametric testing problems and present the Wilks type of results for various models including the nonparametric regression, varying-coefficient models, generalized varying-coefficient models, varying-coefficient partially linear models, additive models, and spectral density estimation. In the same spirit, we propose a GLRT statistic for the conditional mixture copula models as

$$\lambda_T = L(H_1) - L(H_0).$$

Remark 4. Acar et al. (2013) show the asymptotic property of the proposed GLRT statistic in the i.i.d. scenario, that is,

$$r_k \lambda_T \xrightarrow{d} \chi^2_{\mu_T},$$

where $r_k = e_k/m_k$ and $\mu_T = J r_k e_k |\Phi_z|/h$ with J being the number of testing parameters. However, to the best of our knowledge, the GLRT for copula models with time series data has not been studied, and we leave it to future research.

Because the asymptotic properties of the test statistic require further research, we next propose a bootstrap technology to obtain the p-value of the GLRT statistic as follows:

(i). Compute the estimators $\tilde{\xi}(z_t)$ and $\hat{\xi}(z_t)$ by using the same bandwidth h under the null hypothesis H_0 and the alternative hypothesis H_1, respectively. Then, we obtain the GLRT statistic λ_T;

(ii). Generate a sample sequence $\{\mathbf{x}_t^*, z_t^*\}_{t=1}^T$ from the original data $\{\mathbf{X}_t, Z_t\}_{t=1}^T$ using a stationary bootstrap technique as described in Appendix A;
(iii). Obtain the marginal distributions estimates $\{\hat{u}_t^*\}_{t=1}^T$ by Step 1 described in Section 2.2;
(iv). Use the above bootstrap sample to construct the GLRT statistic λ_T^*; and
(v). Repeat Steps (ii)–(iv) S times (say, $S = 1000$) and obtain S values of the statistic λ_T^*. The p-value of the test is the relative frequency of the event $\{\lambda_T^* > \lambda_T\}$ in the S replications of the bootstrap sampling.

3 | NUMERICAL STUDIES

In this section, we investigate the finite-sample performance of our estimation and model selection procedures through a series of numerical studies. For simplicity, we assume the mixture copula model consists of the Clayton, Gumbel, Frank, and Gaussian copulas. They are widely used in empirical studies because they could describe different dependence structures. Specifically, the Clayton copula exhibits strong lower tail dependence, and can well capture cases such as two markets are likely to crash simultaneously. The Gumbel copula shows strong upper tail dependence and can be an appropriate model when two markets are likely to boom together. The Gaussian copula and the Frank copula exhibit symmetric tail dependence.

The working mixture copula model is then formulated as

$$C(u_1, u_2; \boldsymbol{\omega}(z), \boldsymbol{\theta}(z)) = \omega_{Cl}(z)C_{Cl}(u_1, u_2; \theta_{Cl}(z)) + \omega_{\mathrm{Gu}}(z)C_{\mathrm{Gu}}(u_1, u_2; \theta_{\mathrm{Gu}}(z))$$
$$+ \omega_{\mathrm{Fr}}(z)C_{\mathrm{Fr}}(u_1, u_2; \theta_{\mathrm{Fr}}(z)) + \omega_{Ga}(z)C_{Ga}(u_1, u_2; \theta_{Ga}(z)),$$

where $C_{Cl}(\cdot)$, $C_{\mathrm{Gu}}(\cdot)$, $C_{\mathrm{Fr}}(\cdot)$, and $C_{Ga}(\cdot)$ denote the Clayton, Gumbel, Frank, and Gaussian copulas, respectively. Following Abegaz et al. (2012), we generate the covariate z from the truncated normal distribution with mean 0 and variance 9, and then consider four different types of copula parameter function $\theta(z)$ with $z \in [-2, 2]$:

- Model 1: $\theta(z) = 10 - 1.5z^2$;
- Model 2: $\theta(z) = 10 - 0.02z^2 + 0.4z^3$;
- Model 3: $\theta(z) = 3 + z + 2e^{-2z^2}$;
- Model 4: $\theta(z) = 5 + 2\sin(\pi z) + 2e^{-16z^2}$.

For simplicity, we assume that the first marginal distribution u_1 follows the normal distribution $N(1, 0.5)$ and the second marginal distribution u_2 follows the Student's t-distribution with 4 degrees of freedom. For each sample we calculate the estimates $\hat{\theta}$ at 101 equally spaced grid points $z_i = -1.95 + 0.039i$ for $i \in \{0, 1, \dots, 100\}$. Similar to Acar et al. (2011) and Abegaz et al. (2012), we use the local linear fitting with the regular normal kernel. Each simulation is repeated $M = 1000$ times with the sample size $T \in \{200, 500, 1000\}$.

For comparing purposes, besides the proposed conditional mixture copula method (CM), we additionally consider another popular estimation method for mixture copula. Cai & Wang (2014, CW hereafter) propose a copula selection approach via penalized likelihood plus a shrinkage operator, and establish the asymptotic properties of the proposed penalized likelihood estimator. Similar to CM, this method can also select appropriate copula function and estimate the related parameters simultaneously. The main difference is that CW is only applicable to a mixture copula model with constant weights and copula parameters. In this section, we mainly compare the performance of CM with CW when data are indeed generated from conditional mixture copulas. For completeness, we will also investigate CM's performance when the true model is a constant mixture copula.

We begin with a simple scenario that data are generated from an individual copula. That is, the true model is an individual copula selected from the four candidates. For each individual copula used to generate data, we assume the function of the parameter follows one of Models 1–4 listed above. Then, we fit the four-component mixture model to the generated data and investigate the performance of the proposed CM method. In Table 1, we conduct both CM and CW, and report the percentage that each copula is correctly (incorrectly) selected and the mean squared errors (MSEs) of the copula parameter estimates over the 101 grid points, which is defined as

$$\mathrm{MSE}(\hat{\theta}) = \frac{1}{M}\frac{1}{101}\sum_{j=1}^{M}\sum_{i=1}^{101}\left(\hat{\theta}_j(z_i) - \theta(z_i)\right)^2.$$

Table 1 shows that the MSEs of copula parameter estimates by both methods decrease for all four functional forms of $\theta(z)$ as the sample size T increases from 200 to 1000. Here, because the Gaussian copula's parameter θ is ranged between -1 and 1, for Gaussian, we recalibrate the four models by dividing 10 for Model 1 and 15 for Models 2–4. As anticipated, MSEs by CM are remarkably lower than those by CW, indicating larger estimation losses produced by CW when parameters in a copula model are indeed conditioning on a covariate. Besides the estimation accuracy measured by MSE, considering that the parameter functions are assumed to follow Models 1–4 which exhibit different patterns, we additionally examine the quality of the CM estimators by checking their estimated paths along the covariate z. Specifically, we plot the estimated paths of Clayton, Gumbel, Frank, and Gaussian with the sample size $T = 1000$ in Figure 1. The black solid curves in the four panels of each row denote the true copula parameter paths $\theta(z)$, which, respectively, follow Models 1–4, and the other two curves, respectively, denote the means (red dotted) and medians (blue dashed) of the copula parameter estimates by CM at the 101 grid points from 1000 simulations. The two black dotted-dashed curves connect the 5% and 95% percentiles of the copula parameter estimates at the 101 grid points. As a comparison, the mean of the estimated copula parameters by CW from 1000 simulations is also plotted and denoted by the brown solid line. For the four candidate copulas, Figure 1 shows that both the mean and median curves by CM are close to the true paths in all four models. Even in Model 4 which contains the complicated sinus function, the performance of the CM estimator is still quite good. On the other hand, the copula parameter estimated by CW is a constant and therefore cannot detect the dynamics in copula parameters conditioning on the covariate. We also examine the results of model selection through the information criterion method and the CW method. In Table 1, values without parentheses represent rates of correctly selected copulas (accurate rates), while values with parentheses indicate rates that copulas are incorrectly selected (inaccurate rates). One can easily observe that the proposed CM method performs reasonably well in selecting the correct individual copula from the mixture model because the true copula is always chosen with 100% chance, and the rates of incorrect selection shrink when the sample size T increases. CW also exhibits good performance in selecting the true candidate copula. In sum, in terms of parameter estimation accuracy exhibited by MSE and model selection accuracy documented by accurate (inaccurate) rate, the proposed CM method displays excellent performance when the true model is an individual copula. Although CW also exhibits high accuracy in copula selection, it fails to capture the dynamic copula parameters in a conditional copula setup. For completeness, we additionally check the results by the proposed hypothesis test procedure with the 0.05 significance level and find similar results. The detailed simulation results are displayed in Table D4 of Appendix D.

Next, we investigate the performance of CM and CW when the true model is a mixture of two copulas. In other words, for the four-component mixture copula, we assume two candidate copulas' weights uniformly equal to zero while the other two copulas' weights, respectively, equal to $(1+z)^2/29+0.3$ and $1-\left((1+z)^2/29+0.3\right)$. For the two component copulas with nonzero weights, we further assume their parameter functions follow different patterns determined by the four models discussed above. Tables 2 and 3 document the MSEs of copula parameter estimates and the accurate and inaccurate (in parentheses) rates of copula selection by CM and CW, respectively. For example, Panel 1 of Tables 2 and 3 concerns the case that the true mixture model is constructed by Clayton and Gumbel. Comparing results in the two tables, one can observe that the MSEs of the two copulas' parameter estimates by CM decline substantially when the sample size T increases from 200 to 1000, and the magnitudes are remarkably lower than those by CW. In addition, Table 2 indicates that the accurate and inaccurate rates by CM display promising improvement as the sample size increases: in Panel 1, the rates that the Frank and Gaussian copulas are incorrectly selected decrease while the accurate rate for Gumbel increases to about 99% when $T=1000$. There is a 100% probability that the Clayton copula is correctly selected. CW displays similar patterns in copula selection, as can been seen in Table 3. We have similar findings from the other five combinations in both tables. As in the prior individual copula scenario, in Figure 2 we plot the paths of copula parameter estimates, and compare the estimated paths by CM and CW with the true paths. To save space, here we only demonstrate the six combinations of parameter functions in Panel 2 of Tables 2 and 3 when $T=1000$. In Figure 2, the two plots in each column represent a combination of two copulas with different parameter functions. For example, in Figure 2(a), the upper plot demonstrates the true path (Model 1), the mean and median of the estimated paths by CM, and the mean of the estimated path by CW for Clayton, while the lower plot contains the corresponding results for Frank. In general, Figure 2 shows that the copula parameters of the Clayton–Frank mixture can be well estimated by the proposed CM method in all six combinations, while the estimates by CW are constants and unable to detect how copula parameters varies with the covariate.

In addition to copula parameters, it is also worth examining the performance of the proposed CM method in estimating weights of each candidate copula. Using the same four-component mixture model, without loss of generality, we assume the weight of the first copula in the true mixture model follows $(1+z)^2/29+0.3$, and that of the second copula follows $1-\left((1+z)^2/29+0.3\right)$. We display the MSEs of the weight estimates by both CM and CW for all six mixture models in Table 4. As can be seen therein, the MSEs of the weight estimates by both methods decrease in all cases when sample size T increases, and CM uniformly exhibits lower MSEs than CW. Similar to copula parameter estimates, we also draw the true paths, the mean and median of the estimated paths by CM, and the mean of the estimated paths by CW for weight parameters in Figure 3. It corresponds to the weights for the Clayton–Frank combination in Panel 2 of Table 4 when $T=1000$. Figure 3 shows that both the mean and median paths of the weight estimates by CM track the true paths closely, implying a good performance of the proposed CM method in estimating the weights of the Clayton–Frank combination. Weight estimates by CW are constants and thus cannot detect how weights vary with the covariate.

TABLE 1 Mean squared errors (MSEs) of copula parameter estimates and accurate (inaccurate) rates of selection by the proposed conditional mixture copula (CM) and Chen and Fan (2006b) constant mixture copula (CW) when the true model is an individual conditional copula

		True Copula: Clayton																							
		$T = 200$								$T = 500$								$T = 1000$							
		Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian	
Panel 1		CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW
Model 1	MSE	0.731	3.955	–	–	–	–	–	–	0.522	2.770	–	–	–	–	–	–	0.354	1.949	–	–	–	–	–	–
	Rate	1.000	1.000	(0.000)	(0.000)	(0.066)	(0.073)	(0.052)	(0.051)	1.000	1.000	(0.000)	(0.000)	(0.041)	(0.049)	(0.034)	(0.033)	1.000	1.000	(0.000)	(0.000)	(0.025)	(0.027)	(0.011)	(0.014)
Model 2	MSE	0.764	2.783	–	–	–	–	–	–	0.456	1.732	–	–	–	–	–	–	0.252	1.132	–	–	–	–	–	–
	Rate	1.000	1.000	(0.000)	(0.000)	(0.091)	(0.091)	(0.094)	(0.098)	1.000	1.000	(0.000)	(0.000)	(0.049)	(0.053)	(0.058)	(0.061)	1.000	1.000	(0.000)	(0.000)	(0.019)	(0.022)	(0.020)	(0.025)
Model 3	MSE	0.801	3.106	–	–	–	–	–	–	0.437	2.008	–	–	–	–	–	–	0.273	1.576	–	–	–	–	–	–
	Rate	1.000	1.000	(0.000)	(0.000)	(0.064)	(0.092)	(0.061)	(0.079)	1.000	1.000	(0.000)	(0.000)	(0.012)	(0.018)	(0.037)	(0.046)	1.000	1.000	(0.000)	(0.000)	(0.000)	(0.006)	(0.017)	(0.025)
Model 4	MSE	0.714	2.854	–	–	–	–	–	–	0.495	2.154	–	–	–	–	–	–	0.236	1.274	–	–	–	–	–	–
	Rate	1.000	1.000	(0.000)	(0.000)	(0.061)	(0.062)	(0.079)	(0.091)	1.000	1.000	(0.000)	(0.000)	(0.024)	(0.025)	(0.031)	(0.036)	1.000	1.000	(0.000)	(0.000)	(0.011)	(0.014)	(0.009)	(0.014)

		True Copula: Gumbel																							
		$T = 200$								$T = 500$								$T = 1000$							
		Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian	
Panel 2		CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW
Model 1	MSE	–	–	0.753	3.719	–	–	–	–	–	–	0.409	2.481	–	–	–	–	–	–	0.230	1.716	–	–	–	–
	Rate	(0.000)	(0.000)	1.000	1.000	(0.074)	(0.086)	(0.035)	(0.037)	(0.000)	(0.000)	1.000	1.000	(0.043)	(0.054)	(0.021)	(0.024)	(0.000)	(0.000)	1.000	1.000	(0.021)	(0.027)	(0.013)	(0.011)
Model 2	MSE	–	–	0.929	3.004	–	–	–	–	–	–	0.672	2.559	–	–	–	–	–	–	0.329	1.453	–	–	–	–
	Rate	(0.000)	(0.000)	1.000	1.000	(0.084)	(0.094)	(0.085)	(0.096)	(0.000)	(0.000)	1.000	1.000	(0.049)	(0.057)	(0.043)	(0.055)	(0.000)	(0.000)	1.000	1.000	(0.017)	(0.023)	(0.025)	(0.034)
Model 3	MSE	–	–	0.625	2.387	–	–	–	–	–	–	0.433	1.721	–	–	–	–	–	–	0.248	1.385	–	–	–	–
	Rate	(0.000)	(0.000)	1.000	1.000	(0.071)	(0.076)	(0.072)	(0.070)	(0.000)	(0.000)	1.000	1.000	(0.030)	(0.044)	(0.048)	(0.050)	(0.000)	(0.000)	1.000	1.000	(0.010)	(0.011)	(0.019)	(0.011)

(Continues)

TABLE 1 (Continued)

		True Copula: Gumbel																							
		$T = 200$								$T = 500$								$T = 1000$							
		Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian	
Panel 2		CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW
Model 4	MSE	–	–	0.813	3.776	–	–	–	–	–	–	0.441	2.568	–	–	–	–	–	–	0.215	2.070	–	–	–	–
	Rate	(0.000)	(0.000)	1.000	1.000	(0.058)	(0.057)	(0.055)	(0.051)	(0.000)	(0.000)	1.000	1.000	(0.037)	(0.041)	(0.025)	(0.030)	(0.000)	(0.000)	1.000	1.000	(0.019)	(0.021)	(0.008)	(0.012)
		True Copula: Frank																							
		$T = 200$								$T = 500$								$T = 1000$							
		Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian	
Panel 3		CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW
Model 1	MSE	–	–	–	–	0.835	3.555	–	–	–	–	–	–	0.528	2.731	–	–	–	–	–	–	0.231	1.711	–	–
	Rate	(0.000)	(0.000)	(0.011)	(0.012)	1.000	1.000	(0.183)	(0.221)	(0.000)	(0.000)	(0.000)	(0.000)	1.000	1.000	(0.075)	(0.104)	(0.000)	(0.000)	(0.000)	(0.000)	1.000	1.000	(0.026)	(0.038)
Model 2	MSE	–	–	–	–	0.787	2.867	–	–	–	–	–	–	0.429	1.938	–	–	–	–	–	–	0.257	1.454	–	–
	Rate	(0.000)	(0.000)	(0.001)	(0.005)	1.000	1.000	(0.136)	(0.162)	(0.000)	(0.000)	(0.000)	(0.000)	1.000	1.000	(0.054)	(0.088)	(0.000)	(0.000)	(0.000)	(0.000)	1.000	1.000	(0.026)	(0.027)
Model 3	MSE	–	–	–	–	0.820	3.068	–	–	–	–	–	–	0.435	1.920	–	–	–	–	–	–	0.211	1.221	–	–
	Rate	(0.000)	(0.000)	(0.000)	(0.002)	1.000	1.000	(0.219)	(0.220)	(0.000)	(0.000)	(0.000)	(0.000)	1.000	1.000	(0.136)	(0.154)	(0.000)	(0.000)	(0.000)	(0.000)	1.000	1.000	(0.063)	(0.072)
Model 4	MSE	–	–	–	–	0.846	3.695	–	–	–	–	–	–	0.564	2.440	–	–	–	–	–	–	0.226	1.387	–	–
	Rate	(0.000)	(0.000)	(0.000)	(0.000)	1.000	1.000	(0.261)	(0.289)	(0.000)	(0.000)	(0.000)	(0.000)	1.000	1.000	(0.138)	(0.166)	(0.000)	(0.001)	(0.000)	(0.000)	1.000	1.000	(0.058)	(0.067)

(Continues)

TABLE 1 (Continued)

		True Copula: Gaussian																							
		$T = 200$								$T = 500$								$T = 1000$							
		Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian	
Panel 4		CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW
Model 1	MSE	–	–	–	–	–	–	0.011	0.031	–	–	–	–	–	–	0.005	0.021	–	–	–	–	–	–	0.002	0.010
	Rate	(0.000)	(0.000)	(0.000)	(0.000)	(0.107)	(0.176)	1.000	1.000	(0.000)	(0.000)	(0.000)	(0.000)	(0.051)	(0.054)	1.000	1.000	(0.000)	(0.000)	(0.000)	(0.000)	(0.033)	(0.039)	1.000	1.000
Model 2	MSE	–	–	–	–	–	–	0.005	0.018	–	–	–	–	–	–	0.003	0.010	–	–	–	–	–	–	0.001	0.006
	Rate	(0.000)	(0.000)	(0.000)	(0.000)	(0.082)	(0.115)	1.000	1.000	(0.000)	(0.000)	(0.000)	(0.000)	(0.035)	(0.037)	1.000	1.000	(0.000)	(0.000)	(0.000)	(0.000)	(0.021)	(0.021)	1.000	1.000
Model 3	MSE	–	–	–	–	–	–	0.004	0.019	–	–	–	–	–	–	0.002	0.010	–	–	–	–	–	–	0.001	0.005
	Rate	(0.000)	(0.000)	(0.000)	(0.000)	(0.099)	(0.120)	1.000	1.000	(0.000)	(0.000)	(0.000)	(0.000)	(0.051)	(0.076)	1.000	1.000	(0.000)	(0.000)	(0.000)	(0.000)	(0.049)	(0.053)	1.000	1.000
Model 4	MSE	–	–	–	–	–	–	0.005	0.023	–	–	–	–	–	–	0.002	0.012	–	–	–	–	–	–	0.001	0.007
	Rate	(0.000)	(0.000)	(0.000)	(0.000)	(0.049)	(0.088)	1.000	1.000	(0.000)	(0.000)	(0.000)	(0.000)	(0.025)	(0.058)	1.000	1.000	(0.000)	(0.000)	(0.000)	(0.000)	(0.011)	(0.031)	1.000	1.000

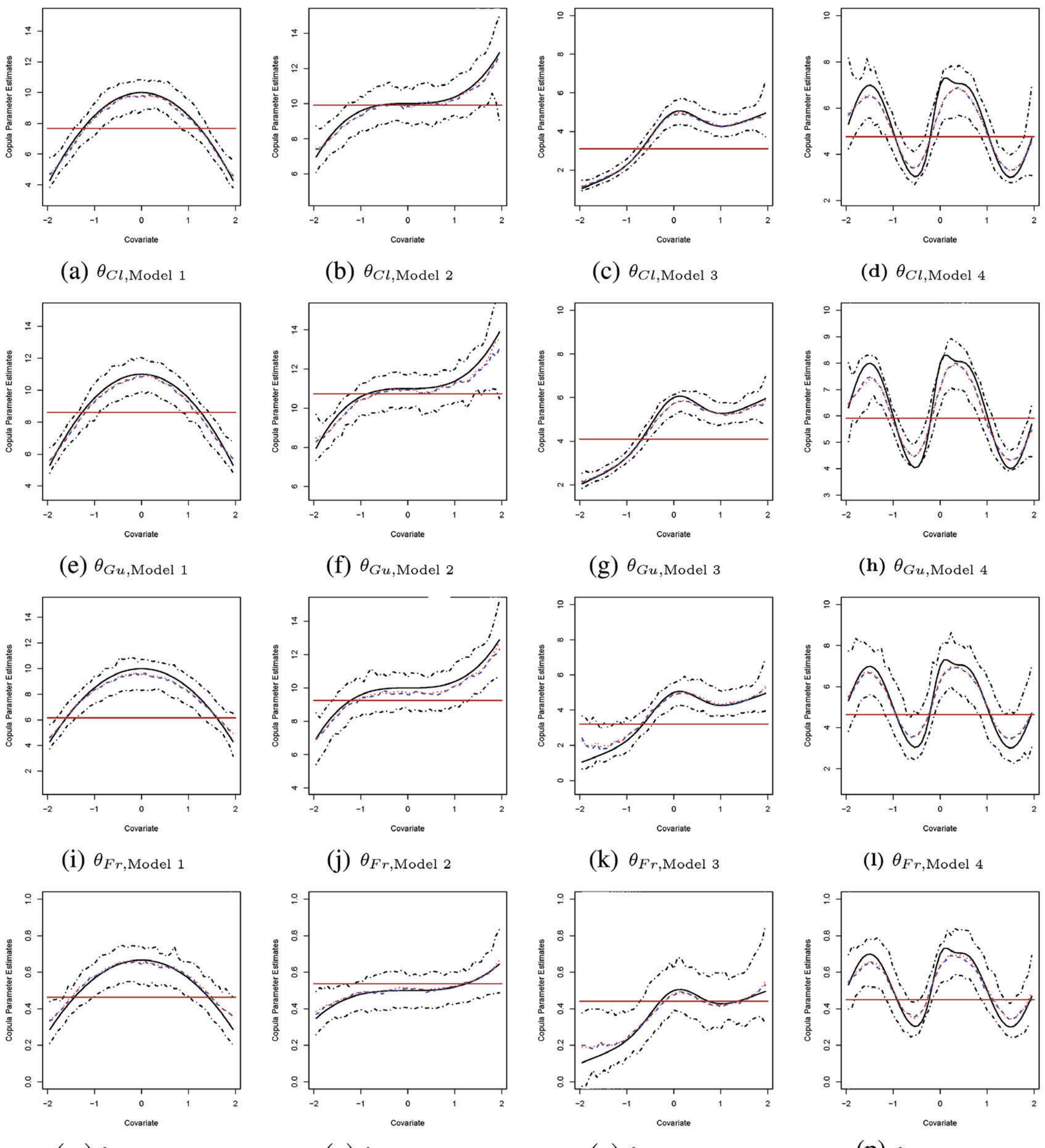


FIGURE 1 Estimated paths for copula parameters (θ) when the true model is an individual copula. *Notes*. Panels (a)–(d) denote the estimates of the Clayton parameters which respectively follow Models 1–4. Panels (e)–(h) denote the estimates of the Gumbel parameters which respectively follow Models 1–4. Panels (i)–(l) denote the estimates of the Frank parameters which respectively follow Models 1–4. Panels (m)–(p) denote the estimates of the Gaussian parameters which respectively follow Models 1-4. Model 1: $\theta(z) = 10 - 1.5z^2$. Model 2: $\theta(z) = 10 - 0.02z^2 + 0.4z^3$. Model 3: $\theta(z) = 3 + z + 2e^{-2z^2}$. Model 4: $\theta(z) = 5 + 2\sin(\pi z) + 2e^{-16z^2}$. In each panel, the black solid line denotes the true path of $\theta(z)$. The red dotted line and the blue dashed line respectively denote the mean and median of the copula parameter function estimates at the grid points with 1000 simulations. The brown solid line denotes the mean of the estimates with 1000 simulations by Garcia and Tsafack (2011). The black dotted-dashed lines denote the 5% and 95% percentiles of the copula parameter estimates at the grid points. The sample size $T = 1000$ in all panels

TABLE 2 Mean squared errors (MSEs) of copula parameter estimates and accurate (inaccurate) rates of selection by the proposed conditional mixture copula (CM) when the true model is a conditional mixture copula

		Combination: Clayton + Gumbel											
		$T=200$				$T=500$				$T=1000$			
Panel 1		**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**
Model 1 + Model 2	MSE	0.903	0.850	–	–	0.633	0.521	–	–	0.303	0.287	–	–
	Rate	1.000	0.859	(0.151)	(0.234)	1.000	0.936	(0.064)	(0.125)	1.000	0.986	(0.012)	(0.053)
Model 1 + Model 3	MSE	0.893	0.775	–	–	0.472	0.405	–	–	0.288	0.197	–	–
	Rate	1.000	0.901	(0.078)	(0.212)	1.000	0.927	(0.022)	(0.101)	1.000	0.973	(0.007)	(0.061)
Model 1 + Model 4	MSE	0.941	0.901	–	–	0.507	0.476	–	–	0.273	0.286	–	–
	Rate	1.000	0.925	(0.039)	(0.277)	1.000	0.941	(0.011)	(0.121)	1.000	0.991	(0.003)	(0.067)
Model 2 + Model 3	MSE	1.032	0.957	–	–	0.688	0.530	–	–	0.301	0.251	–	–
	Rate	1.000	0.863	(0.136)	(0.209)	1.000	0.923	(0.034)	(0.091)	1.000	0.985	(0.016)	(0.044)
Model 2 + Model 4	MSE	1.005	0.833	–	–	0.605	0.497	–	–	0.331	0.244	–	–
	Rate	1.000	0.874	(0.130)	(0.291)	1.000	0.945	(0.031)	(0.139)	1.000	0.987	(0.005)	(0.078)
Model 3 + Model 4	MSE	0.843	0.860	–	–	0.512	0.490	–	–	0.346	0.268	–	–
	Rate	1.000	0.869	(0.142)	(0.226)	1.000	0.954	(0.047)	(0.116)	1.000	0.994	(0.008)	(0.047)
		Combination: Clayton + Frank											
		$T=200$				$T=500$				$T=1000$			
Panel 2		**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**
Model 1 + Model 2	MSE	1.036	–	0.925	–	0.633	–	0.447	–	0.380	–	0.251	–
	Rate	1.000	(0.015)	0.874	(0.183)	1.000	(0.003)	0.908	(0.098)	1.000	(0.000)	0.993	(0.043)
Model 1 + Model 3	MSE	1.047	–	0.859	–	0.629	–	0.517	–	0.291	–	0.266	–
	Rate	1.000	(0.022)	0.891	(0.194)	1.000	(0.006)	0.939	(0.090)	1.000	(0.000)	0.997	(0.036)

(Continues)

TABLE 2 (Continued)

		Combination: Clayton + Frank											
		$T = 200$				$T = 500$				$T = 1000$			
Panel 2		**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**
Model 1 + Model 4	MSE	0.931	–	0.906	–	0.424	–	0.601	–	0.230	–	0.325	–
	Rate	1.000	(0.017)	0.923	(0.207)	1.000	(0.000)	0.971	(0.107)	1.000	(0.000)	1.000	(0.029)
Model 2 + Model 3	MSE	0.935	–	0.923	–	0.591	–	0.459	–	0.372	–	0.245	–
	Rate	1.000	(0.021)	0.856	(0.251)	1.000	(0.000)	0.911	(0.137)	1.000	(0.000)	0.984	(0.061)
Model 2 + Model 4	MSE	0.817	–	0.852	–	0.456	–	0.427	–	0.296	–	0.289	–
	Rate	1.000	(0.014)	0.904	(0.191)	1.000	(0.003)	0.975	(0.112)	1.000	(0.000)	0.994	(0.054)
Model 3 + Model 4	MSE	0.831	–	0.874	–	0.533	–	0.512	–	0.230	–	0.222	–
	Rate	1.000	(0.009)	0.899	(0.233)	1.000	(0.000)	0.964	(0.128)	1.000	(0.000)	1.000	(0.072)
		Combination: Gumbel + Frank											
		$T = 200$				$T = 500$				$T = 1000$			
Panel 3		**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**
Model 1 + Model 2	MSE	–	0.814	0.933	–	–	0.513	0.481	–	–	0.289	0.227	–
	Rate	(0.003)	0.975	0.874	(0.151)	(0.001)	0.987	0.957	(0.078)	(0.000)	0.998	0.983	(0.023)
Model 1 + Model 3	MSE	–	0.853	0.891	–	–	0.503	0.465	–	–	0.275	0.201	–
	Rate	(0.000)	0.982	0.901	(0.187)	(0.000)	0.986	0.989	(0.084)	(0.000)	0.993	0.981	(0.051)
Model 1 + Model 4	MSE	–	0.930	0.942	–	–	0.535	0.526	–	–	0.208	0.267	–
	Rate	(0.004)	0.974	0.887	(0.174)	(0.000)	0.983	0.946	(0.081)	(0.000)	1.000	0.992	(0.038)
Model 2 + Model 3	MSE	–	0.727	0.865	–	–	0.517	0.574	–	–	0.305	0.264	–
	Rate	(0.002)	0.986	0.939	(0.152)	(0.000)	0.977	0.951	(0.073)	(0.000)	0.997	0.998	(0.047)

(Continues)

TABLE 2 (Continued)

		Combination: Gumbel + Frank											
		$T = 200$				$T = 500$				$T = 1000$			
Panel 3		**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**
Model 2 + Model 4	MSE	–	0.958	0.771	–	–	0.466	0.397	–	–	0.268	0.203	–
	Rate	(0.001)	0.967	0.853	(0.131)	(0.000)	0.981	0.976	(0.062)	(0.000)	1.000	0.993	(0.013)
Model 3 + Model 4	MSE	–	0.783	0.732	–	–	0.463	0.402	–	–	0.326	0.340	–
	Rate	(0.000)	0.958	0.944	(0.147)	(0.001)	0.967	0.983	(0.064)	(0.000)	0.987	0.987	(0.022)
		True Copula: Clayton + Gaussian											
		$T = 200$				$T = 500$				$T = 1000$			
Panel 4		**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**
Model 1 + Model 2	MSE	0.707	–	–	0.005	0.411	–	–	0.003	0.231	–	–	0.001
	Rate	1.000	(0.011)	(0.193)	0.961	1.000	(0.004)	(0.081)	0.998	1.000	(0.000)	(0.042)	1.000
Model 1 + Model 3	MSE	0.735	–	–	0.007	0.398	–	–	0.004	0.250	–	–	0.003
	Rate	1.000	(0.005)	(0.152)	0.954	1.000	(0.001)	(0.078)	0.972	1.000	(0.000)	(0.039)	0.995
Model 1 + Model 4	MSE	0.852	–	–	0.008	0.501	–	–	0.004	0.222	–	–	0.001
	Rate	1.000	(0.011)	(0.183)	0.957	1.000	(0.000)	(0.075)	0.984	1.000	(0.000)	(0.037)	1.000
Model 2 + Model 3	MSE	0.753	–	–	0.006	0.443	–	–	0.004	0.205	–	–	0.002
	Rate	1.000	(0.016)	(0.179)	0.947	1.000	(0.001)	(0.083)	0.988	1.000	(0.000)	(0.041)	1.000
Model 2 + Model 4	MSE	0.849	–	–	0.005	0.529	–	–	0.003	0.230	–	–	0.001
	Rate	1.000	(0.021)	(0.182)	0.957	1.000	(0.003)	(0.086)	0.997	1.000	(0.000)	(0.047)	1.000
Model 3 + Model 4	MSE	0.831	–	–	0.006	0.477	–	–	0.003	0.209	–	–	0.001
	Rate	1.000	(0.008)	(0.176)	0.944	1.000	(0.000)	(0.081)	0.989	1.000	(0.000)	(0.039)	1.000

(Continues)

TABLE 2 (Continued)

		Combination: Gumbel + Gaussian											
		$T = 200$				$T = 500$				$T = 1000$			
Panel 5		**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**
Model 1 + Model 2	MSE	–	0.883	–	0.008	–	0.474	–	0.005	–	0.208	–	0.002
	Rate	(0.000)	0.931	(0.245)	0.964	(0.000)	0.974	(0.121)	1.000	(0.000)	0.998	(0.052)	1.000
Model 1 + Model 3	MSE	–	0.920	–	0.007	–	0.448	–	0.005	–	0.233	–	0.002
	Rate	(0.000)	0.945	(0.196)	0.947	(0.000)	0.961	(0.089)	0.995	(0.000)	0.994	(0.037)	1.000
Model 1 + Model 4	MSE	–	0.844	–	0.008	–	0.503	–	0.006	–	0.220	–	0.003
	Rate	(0.000)	0.927	(0.203)	0.953	(0.000)	0.979	(0.117)	0.989	(0.000)	0.987	(0.066)	0.994
Model 2 + Model 3	MSE	–	0.797	–	0.008	–	0.427	–	0.005	–	0.269	–	0.002
	Rate	(0.000)	0.958	(0.192)	0.961	(0.000)	0.978	(0.102)	0.993	(0.000)	0.993	(0.051)	1.000
Model 2 + Model 4	MSE	–	0.835	–	0.006	–	0.474	–	0.004	–	0.283	–	0.002
	Rate	(0.000)	0.944	(0.235)	0.955	(0.000)	0.980	(0.115)	0.987	(0.000)	0.998	(0.064)	1.000
Model 3 + Model 4	MSE	–	0.937	–	0.007	–	0.539	–	0.004	–	0.317	–	0.002
	Rate	(0.000)	0.936	(0.197)	0.941	(0.000)	0.979	(0.099)	0.979	(0.000)	0.987	(0.033)	0.998

(Continues)

TABLE 2 (Continued)

Panel 6		Combination: Frank + Gaussian											
		$T = 200$				$T = 500$				$T = 1000$			
		Clayton	Gumbel	Frank	Gaussian	Clayton	Gumbel	Frank	Gaussian	Clayton	Gumbel	Frank	Gaussian
Model 1 + Model 2	MSE	–	–	0.879	0.011	–	–	0.512	0.008	–	–	0.274	0.003
	Rate	(0.000)	(0.011)	0.875	0.983	(0.000)	(0.000)	0.913	1.000	(0.000)	(0.000)	0.983	1.000
Model 1 + Model 3	MSE	–	–	0.742	0.009	–	–	0.442	0.006	–	–	0.260	0.002
	Rate	(0.000)	(0.013)	0.854	0.987	(0.000)	(0.000)	0.907	1.000	(0.000)	(0.000)	0.992	1.000
Model 1 + Model 4	MSE	–	–	0.833	0.013	–	–	0.435	0.009	–	–	0.279	0.005
	Rate	(0.000)	(0.015)	0.886	0.991	(0.000)	(0.000)	0.914	1.000	(0.000)	(0.000)	0.985	1.000
Model 2 + Model 3	MSE	–	–	0.858	0.008	–	–	0.408	0.005	–	–	0.251	0.003
	Rate	(0.000)	(0.012)	0.873	0.989	(0.000)	(0.000)	0.905	0.993	(0.000)	(0.000)	0.990	1.000
Model 2 + Model 4	MSE	–	–	0.903	0.011	–	–	0.511	0.006	–	–	0.281	0.003
	Rate	(0.000)	(0.011)	0.862	0.991	(0.000)	(0.000)	0.916	1.000	(0.000)	(0.000)	0.989	1.000
Model 3 + Model 4	MSE	–	–	0.944	0.013	–	–	0.553	0.007	–	–	0.302	0.003
	Rate	(0.000)	(0.015)	0.887	0.983	(0.000)	(0.000)	0.920	1.000	(0.000)	(0.000)	0.983	1.000

TABLE 3 Mean squared errors (MSEs) of copula parameter estimates and accurate (inaccurate) rates of selection by Longin & Solnik (2001)and Garcia and Tsafack (2011) constant mixture copula (CW) when the true model is a conditional mixture copula

		Combination: Clayton + Gumbel											
		$T = 200$				$T = 500$				$T = 1000$			
Panel 1		**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**
Model 1 + Model 2	MSE	4.836	3.129	–	–	3.166	2.646	–	–	2.401	1.480	–	–
	Rate	1.000	0.931	(0.174)	(0.254)	1.000	0.955	(0.106)	(0.183)	1.000	0.993	(0.033)	(0.104)
Model 1 + Model 3	MSE	3.991	3.528	–	–	2.597	2.445	–	–	1.931	1.558	–	–
	Rate	1.000	0.921	(0.112)	(0.287)	1.000	0.948	(0.054)	(0.113)	1.000	0.984	(0.013)	(0.065)
Model 1 + Model 4	MSE	3.821	3.681	–	–	2.638	2.353	–	–	1.514	1.231	–	–
	Rate	1.000	0.930	(0.104)	(0.296)	1.000	0.964	(0.035)	(0.152)	1.000	1.000	(0.007)	(0.077)
Model 2 + Model 3	MSE	4.658	4.436	–	–	2.701	2.411	–	–	1.512	1.382	–	–
	Rate	1.000	0.916	(0.158)	(0.253)	1.000	0.958	(0.049)	(0.093)	1.000	0.989	(0.014)	(0.039)
Model 2 + Model 4	MSE	4.819	4.627	–	–	2.706	2.568	–	–	1.705	1.588	–	–
	Rate	1.000	0.948	(0.129)	(0.301)	1.000	0.962	(0.034)	(0.154)	1.000	1.000	(0.015)	(0.088)
Model 3 + Model 4	MSE	3.158	3.687	–	–	2.546	2.610	–	–	1.447	1.658	–	–
	Rate	1.000	0.905	(0.157)	(0.258)	1.000	0.932	(0.079)	(0.128)	1.000	1.000	(0.019)	(0.041)
		Combination: Clayton + Frank											
		$T = 200$				$T = 500$				$T = 1000$			
Panel 2		**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**
Model 1 + Model 2	MSE	3.769	–	3.197	–	2.477	–	2.653	–	1.296	–	1.424	–
	Rate	1.000	(0.023)	0.931	(0.205)	1.000	(0.005)	0.923	(0.134)	1.000	(0.000)	1.000	(0.072)

(Continues)

TABLE 3 (Continued)

Panel 2		Clayton	Gumbel	Frank	Gaussian	Clayton	Gumbel	Frank	Gaussian	Clayton	Gumbel	Frank	Gaussian
		Combination: Clayton + Frank											
		$T = 200$				$T = 500$				$T = 1000$			
Model 1 + Model 3	MSE	3.825	–	3.003	–	2.402	–	1.847	–	1.838	–	1.097	–
	Rate	1.000	(0.025)	0.925	(0.237)	1.000	(0.000)	0.944	(0.118)	1.000	(0.000)	1.000	(0.063)
Model 1 + Model 4	MSE	3.770	–	3.568	–	2.581	–	2.419	–	1.403	–	1.370	–
	Rate	1.000	(0.020)	0.895	(0.262)	1.000	(0.003)	0.946	(0.120)	1.000	(0.000)	1.000	(0.055)
Model 2 + Model 3	MSE	3.704	–	3.239	–	2.688	–	2.172	–	1.522	–	1.165	–
	Rate	1.000	(0.018)	0.907	(0.289)	1.000	(0.001)	0.957	(0.154)	1.000	(0.000)	1.000	(0.069)
Model 2 + Model 4	MSE	2.911	–	2.941	–	2.249	–	2.447	–	1.239	–	1.384	–
	Rate	1.000	(0.015)	0.922	(0.264)	1.000	(0.003)	0.955	(0.116)	1.000	(0.000)	1.000	(0.048)
Model 3 + Model 4	MSE	3.335	–	3.569	–	2.137	–	2.398	–	1.393	–	1.371	–
	Rate	1.000	(0.013)	0.873	(0.262)	1.000	(0.000)	0.931	(0.173)	1.000	(0.000)	0.998	(0.087)
		Combination: Gumbel + Frank											
		$T = 200$				$T = 500$				$T = 1000$			
Panel 3		**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**
Model 1 + Model 2	MSE	–	3.584	3.276	–	–	2.473	2.443	–	–	1.397	1.266	–
	Rate	(0.005)	0.985	0.903	(0.187)	(0.001)	1.000	0.992	(0.085)	(0.000)	1.000	0.995	(0.031)
Model 1 + Model 3	MSE	–	3.747	3.255	–	–	2.465	2.677	–	–	1.378	1.523	–
	Rate	(0.000)	0.957	0.942	(0.195)	(0.000)	0.994	1.000	(0.082)	(0.000)	1.000	1.000	(0.050)
Model 1 + Model 4	MSE	–	3.726	3.580	–	–	2.461	2.062	–	–	1.385	1.037	–
	Rate	(0.005)	0.989	0.896	(0.193)	(0.000)	0.990	0.977	(0.088)	(0.000)	1.000	1.000	(0.045)

(Continues)

TABLE 3 (Continued)

		Combination: Gumbel + Frank											
		$T = 200$				$T = 500$				$T = 1000$			
Panel 3		Clayton	Gumbel	Frank	Gaussian	Clayton	Gumbel	Frank	Gaussian	Clayton	Gumbel	Frank	Gaussian
Model 2 + Model 3	MSE	–	3.105	3.175	–	–	2.562	2.403	–	–	1.502	1.143	–
	Rate	(0.001)	0.971	0.944	(0.188)	(0.000)	0.991	0.984	(0.073)	(0.000)	1.000	1.000	(0.017)
Model 2 + Model 4	MSE	–	3.716	3.528	–	–	2.527	2.508	–	–	1.468	1.395	–
	Rate	(0.000)	0.963	0.907	(0.172)	(0.000)	1.000	0.997	(0.089)	(0.000)	1.000	1.000	(0.013)
Model 3 + Model 4	MSE	–	3.454	3.721	–	–	2.409	2.116	–	–	1.443	1.222	–
	Rate	(0.002)	0.970	0.905	(0.133)	(0.001)	0.987	1.000	(0.061)	(0.000)	0.998	1.000	(0.024)
		Combination: Clayton + Gaussian											
		$T = 200$				$T = 500$				$T = 1000$			
Panel 4		Clayton	Gumbel	Frank	Gaussian	Clayton	Gumbel	Frank	Gaussian	Clayton	Gumbel	Frank	Gaussian
Model 1 + Model 2	MSE	3.894	–	–	0.027	2.502	–	–	0.016	1.299	–	–	0.005
	Rate	1.000	(0.017)	(0.227)	0.957	1.000	(0.008)	(0.103)	1.000	1.000	(0.000)	(0.034)	1.000
Model 1 + Model 3	MSE	3.780	–	–	0.032	3.495	–	–	0.019	3.306	–	–	0.009
	Rate	1.000	(0.004)	(0.204)	0.974	1.000	(0.000)	(0.094)	0.986	1.000	(0.000)	(0.062)	1.000
Model 1 + Model 4	MSE	3.827	–	–	0.044	3.553	–	–	0.022	3.269	–	–	0.008
	Rate	1.000	(0.016)	(0.176)	0.977	1.000	(0.000)	(0.068)	0.982	1.000	(0.000)	(0.049)	1.000
Model 2 + Model 3	MSE	2.820	–	–	0.023	1.849	–	–	0.013	1.031	–	–	0.006
	Rate	1.000	(0.015)	(0.213)	0.965	1.000	(0.000)	(0.091)	0.975	1.000	(0.000)	(0.063)	1.000
Model 2 + Model 4	MSE	2.917	–	–	0.023	2.041	–	–	0.011	1.665	–	–	0.005
	Rate	1.000	(0.029)	(0.204)	0.973	1.000	(0.005)	(0.085)	0.984	1.000	(0.000)	(0.059)	0.997

(Continues)

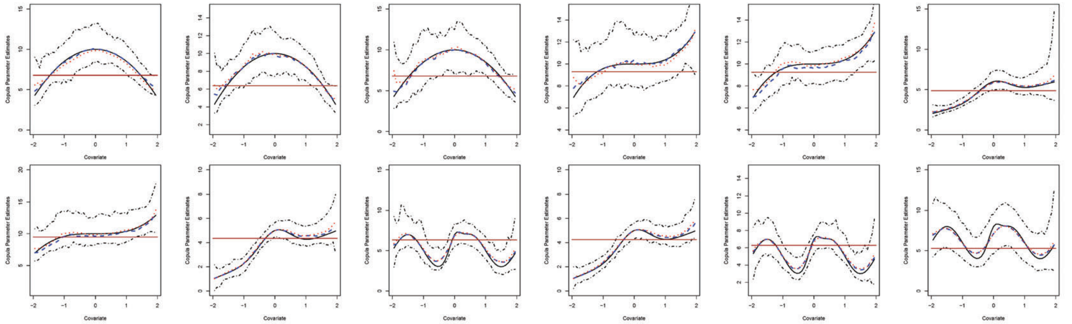
TABLE 3 (Continued)

		Combination: Clayton + Gaussian											
		$T = 200$				$T = 500$				$T = 1000$			
Panel 4		**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**
Model 3 + Model 4	MSE	3.606	–	–	0.035	2.563	–	–	0.012	1.717	–	–	0.004
	Rate	1.000	(0.005)	(0.198)	0.968	1.000	(0.002)	(0.094)	0.970	1.000	(0.000)	(0.045)	1.000
		Combination: Gumbel + Gaussian											
		$T = 200$				$T = 500$				$T = 1000$			
Panel 5		**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**
Model 1 + Model 2	MSE	–	3.548	–	0.048	–	2.517	–	0.027	–	1.422	–	0.008
	Rate	(0.000)	0.944	(0.278)	0.974	(0.000)	0.988	(0.133)	1.000	(0.000)	1.000	(0.048)	1.000
Model 1 + Model 3	MSE	–	3.569	–	0.033	–	2.562	–	0.020	–	1.835	–	0.006
	Rate	(0.000)	0.941	(0.212)	0.931	(0.000)	0.955	(0.120)	1.000	(0.000)	1.000	(0.046)	1.000
Model 1 + Model 4	MSE	–	3.730	–	0.044	–	2.377	–	0.032	–	1.348	–	0.009
	Rate	(0.000)	0.908	(0.174)	0.948	(0.000)	0.976	(0.119)	1.000	(0.000)	0.997	(0.057)	1.000
Model 2 + Model 3	MSE	–	3.026	–	0.042	–	2.220	–	0.019	–	1.534	–	0.009
	Rate	(0.000)	0.923	(0.228)	0.977	(0.000)	0.983	(0.125)	1.000	(0.000)	1.000	(0.061)	1.000
Model 2 + Model 4	MSE	–	4.082	–	0.024	–	2.626	–	0.011	–	1.488	–	0.007
	Rate	(0.000)	0.911	(0.293)	0.934	(0.000)	0.987	(0.127)	0.998	(0.000)	1.000	(0.073)	1.000
Model 3 + Model 4	MSE	–	3.430	–	0.035	–	2.435	–	0.017	–	1.415	–	0.010
	Rate	(0.000)	0.950	(0.204)	0.950	(0.000)	0.959	(0.110)	1.000	(0.000)	1.000	(0.041)	1.000

(Continues)

TABLE 3 (Continued)

		Combination: Frank + Gaussian											
		$T = 200$				$T = 500$				$T = 1000$			
Panel 6		**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**	**Clayton**	**Gumbel**	**Frank**	**Gaussian**
Model 1 + Model 2	MSE	–	–	3.625	0.059	–	–	2.382	0.036	–	–	1.217	0.015
	Rate	(0.000)	(0.016)	0.891	0.974	(0.000)	(0.000)	0.932	1.000	(0.000)	(0.000)	0.998	1.000
Model 1 + Model 3	MSE	–	–	3.770	0.043	–	–	2.587	0.030	–	–	1.226	0.008
	Rate	(0.000)	(0.014)	0.863	0.954	(0.000)	(0.000)	0.894	1.000	(0.000)	(0.000)	1.000	1.000
Model 1 + Model 4	MSE	–	–	3.445	0.065	–	–	2.282	0.042	–	–	1.102	0.020
	Rate	(0.000)	(0.010)	0.895	0.987	(0.000)	(0.000)	0.907	1.000	(0.000)	(0.000)	0.991	1.000
Model 2 + Model 3	MSE	–	–	4.311	0.043	–	–	2.730	0.029	–	–	1.755	0.008
	Rate	(0.000)	(0.017)	0.914	0.980	(0.000)	(0.000)	0.913	1.000	(0.000)	(0.000)	0.986	1.000
Model 2 + Model 4	MSE	–	–	4.430	0.054	–	–	2.993	0.030	–	–	1.944	0.010
	Rate	(0.000)	(0.014)	0.882	0.976	(0.000)	(0.000)	0.896	1.000	(0.000)	(0.000)	0.993	1.000
Model 3 + Model 4	MSE	–	–	3.709	0.064	–	–	2.443	0.031	–	–	1.675	0.009
	Rate	(0.000)	(0.012)	0.854	0.963	(0.000)	(0.000)	0.919	1.000	(0.000)	(0.000)	0.980	1.000



(a) $\theta_{Cl,\text{Mod }1}, \theta_{Fr,\text{Mod }2}$ (b) $\theta_{Cl,\text{Mod }1}, \theta_{Fr,\text{Mod }3}$ (c) $\theta_{Cl,\text{Mod }1}, \theta_{Fr,\text{Mod }4}$ (d) $\theta_{Cl,\text{Mod }2}, \theta_{Fr,\text{Mod }3}$ (e) $\theta_{Cl,\text{Mod }2}, \theta_{Fr,\text{Mod }4}$ (f) $\theta_{Cl,\text{Mod }3}, \theta_{Fr,\text{Mod }4}$

FIGURE 2 Estimated paths for copula parameters ($\boldsymbol{\theta}$) when the true model is a combination of the Clayton and Frank copulas. *Notes*. Panel (a) displays the parameter estimates for the Clayton (up) and Frank (down) copulas when θ_{Cl} = Model 1 and θ_{Fr} = Model 2. Panel (b) displays the parameter estimates for the Clayton (up) and Frank (down) copulas when θ_{Cl} = Model 1 and θ_{Fr} = Model 3. Panel (c) displays the parameter estimates for the Clayton (up) and Frank (down) copulas when θ_{Cl} = Model 1 and θ_{Fr} = Model 4. Panel (d) displays the parameter estimates for the Clayton (up) and Frank (down) copulas when θ_{Cl} = Model 2 and θ_{Fr} = Model 3. Panel (e) displays the parameter estimates for the Clayton (up) and Frank (down) copulas when θ_{Cl} = Model 2 and θ_{Fr} = Model 4. Panel (f) displays the parameter estimates for the Clayton (up) and Frank (down) copulas when θ_{Cl} = Model 3 and θ_{Fr} = Model 4. Model 1: $\theta(z) = 10 - 1.5z^2$. Model 2: $\theta(z) = 10 - 0.02z^2 + 0.4z^3$. Model 3: $\theta(z) = 3 + z + 2e^{-2z^2}$. Model 4: $\theta(z) = 5 + 2\sin(\pi z) + 2e^{-16z^2}$. The black solid line denotes the true path of $\theta(z)$. The red dotted line and the blue dashed line respectively denote the mean and median of the copula parameter function estimates at the grid points with 1000 simulations. The brown solid line denotes the mean of the estimates with 1000 simulations by Garcia and Tsafack (2011). The black dotted-dashed lines denote the 5% and 95% percentiles of the copula parameter estimates at the grid points. The sample size $T = 1000$ in all panels

Finally, we compare the performance of CM and CW when the true mixture copula model exhibits constant parameters. To save space, we only consider two scenarios. First, we assume data are generated from an individual Clayton copula with the dependence parameter equals either 5 or 7. Second, we generate data from a combination of the Clayton and Gumbel copulas. For simplicity, we assume the two copulas are equally weighted with two pairs of constant copula parameters, ($\theta_{\text{Cl}} = 5, \theta_{\text{Gu}} = 4$) and ($\theta_{\text{Cl}} = 7, \theta_{\text{Gu}} = 6$). Table 5 shows that, when the true copula model exhibits constant parameters, the CW method exhibits better performance than the proposed CM method because the MSEs produced by CW are slightly lower than those by CM. This should be expected because CW exhibits higher estimation efficiency when parameters in a mixture copula are indeed constant. In terms of picking up the correct copula functions, both methods exhibit similar performance.

We additionally conduct simulations to investigate the performance of our method when: (i) the conditional mixture model contains three- and four-dimensional copulas, and (ii) data are generated from copulas not included in the candidate set (i.e., the mixture copula is misspecified). These additional simulation results, displayed in Appendix D, provide further evidence that the proposed CM method still displays good performance in the two scenarios.

TABLE 4 Mean squared errors (MSEs) of weight estimates by the proposed conditional mixture copula (CM) and Acar et al. (2011), Patton (2012), and Fermanian and Lopez (2018) constant mixture copula (CW) when the true model is a conditional mixture copula

	True Copula: Clayton + Gumbel																							
	$T = 200$								$T = 500$								$T = 1000$							
	Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian	
Panel 1	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW
Model 1 + Model 2	0.009	0.085	0.010	0.091	–	–	–	–	0.005	0.056	0.009	0.057	–	–	–	–	0.002	0.031	0.003	0.035	–	–	–	–
Model 1 + Model 3	0.010	0.087	0.011	0.084	–	–	–	–	0.006	0.048	0.008	0.050	–	–	–	–	0.002	0.024	0.001	0.021	–	–	–	–
Model 1 + Model 4	0.011	0.091	0.012	0.086	–	–	–	–	0.004	0.055	0.009	0.049	–	–	–	–	0.001	0.029	0.003	0.026	–	–	–	–
Model 2 + Model 3	0.009	0.078	0.007	0.083	–	–	–	–	0.004	0.048	0.005	0.041	–	–	–	–	0.001	0.023	0.002	0.019	–	–	–	–
Model 2 + Model 4	0.013	0.084	0.010	0.078	–	–	–	–	0.006	0.040	0.004	0.047	–	–	–	–	0.002	0.026	0.001	0.021	–	–	–	–
Model 3 + Model 4	0.011	0.086	0.013	0.087	–	–	–	–	0.006	0.041	0.007	0.055	–	–	–	–	0.001	0.019	0.002	0.022	–	–	–	–
	True Copula: Clayton + Frank																							
	$T = 200$								$T = 500$								$T = 1000$							
	Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian	
Panel 2	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW
Model 1 + Model 2	0.013	0.112	–	–	0.009	0.107	–	–	0.007	0.071	–	–	0.004	0.066	–	–	0.002	0.048	–	–	0.002	0.034	–	–
Model 1 + Model 3	0.012	0.088	–	–	0.008	0.116	–	–	0.007	0.052	–	–	0.003	0.063	–	–	0.003	0.039	–	–	0.002	0.040	–	–
Model 1 + Model 4	0.012	0.093	–	–	0.010	0.102	–	–	0.006	0.058	–	–	0.004	0.059	–	–	0.004	0.037	–	–	0.001	0.031	–	–
Model 2 + Model 3	0.010	0.091	–	–	0.011	0.113	–	–	0.003	0.057	–	–	0.006	0.053	–	–	0.001	0.030	–	–	0.004	0.027	–	–

(Continues)

TABLE 4 (Continued)

	True Copula: Clayton + Frank																							
	$T = 200$								$T = 500$								$T = 1000$							
	Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian	
Panel 2	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW
Model 2 + Model 4	0.012	0.102	–	–	0.009	0.098	–	–	0.005	0.054	–	–	0.004	0.061	–	–	0.003	0.026	–	–	0.002	0.035	–	–
Model 3 + Model 4	0.010	0.118	–	–	0.009	0.086	–	–	0.003	0.062	–	–	0.005	0.050	–	–	0.002	0.032	–	–	0.002	0.028	–	–
	True Copula: Gumbel + Frank																							
	$T = 200$								$T = 500$								$T = 1000$							
	Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian	
Panel 3	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW
Model 1 + Model 2	–	–	0.011	0.104	0.009	0.117	–	–	–	–	0.006	0.063	0.004	0.064	–	–	–	–	0.002	0.036	0.002	0.033	–	–
Model 1 + Model 3	–	–	0.011	0.105	0.008	0.106	–	–	–	–	0.007	0.066	0.005	0.062	–	–	–	–	0.003	0.038	0.003	0.036	–	–
Model 1 + Model 4	–	–	0.014	0.097	0.009	0.093	–	–	–	–	0.007	0.057	0.005	0.053	–	–	–	–	0.002	0.035	0.002	0.030	–	–
Model 2 + Model 3	–	–	0.015	0.099	0.008	0.094	–	–	–	–	0.008	0.069	0.003	0.051	–	–	–	–	0.003	0.034	0.002	0.039	–	–
Model 2 + Model 4	–	–	0.009	0.081	0.011	0.109	–	–	–	–	0.005	0.045	0.005	0.050	–	–	–	–	0.001	0.031	0.003	0.032	–	–
Model 3 + Model 4	–	–	0.010	0.093	0.011	0.104	–	–	–	–	0.005	0.045	0.006	0.047	–	–	–	–	0.002	0.035	0.004	0.030	–	–
	True Copula: Clayton + Gaussian																							
	$T = 200$								$T = 500$								$T = 1000$							
	Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian	
Panel 4	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW
Model 1 + Model 2	0.014	0.106	–	–	–	–	0.013	0.098	0.006	0.063	–	–	–	–	0.006	0.059	0.003	0.032	–	–	–	–	0.003	0.036
Model 1 + Model 3	0.013	0.097	–	–	–	–	0.011	0.072	0.007	0.057	–	–	–	–	0.008	0.044	0.004	0.029	–	–	–	–	0.005	0.037

(Continues)

TABLE 4 (Continued)

	True Copula: Clayton + Gaussian																							
	$T = 200$								$T = 500$								$T = 1000$							
	Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian	
Panel 4	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW
Model 1 + Model 4	0.011	0.103	–	–	–	–	0.012	0.069	0.006	0.056	–	–	–	–	0.007	0.042	0.002	0.032	–	–	–	–	0.004	0.031
Model 2 + Model 3	0.013	0.094	–	–	–	–	0.015	0.083	0.005	0.051	–	–	–	–	0.007	0.041	0.003	0.027	–	–	–	–	0.005	0.033
Model 2 + Model 4	0.009	0.095	–	–	–	–	0.014	0.102	0.005	0.053	–	–	–	–	0.008	0.052	0.003	0.029	–	–	–	–	0.004	0.035
Model 3 + Model 4	0.012	0.102	–	–	–	–	0.010	0.095	0.007	0.052	–	–	–	–	0.004	0.045	0.004	0.033	–	–	–	–	0.003	0.034
	True Copula: Gumbel + Gaussian																							
	$T = 200$								$T = 500$								$T = 1000$							
	Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian	
Panel 5	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW
Model 1 + Model 2	–	–	0.007	0.093	–	–	0.008	0.069	–	–	0.004	0.049	–	–	0.004	0.031	–	–	0.002	0.032	–	–	0.002	0.017
Model 1 + Model 3	–	–	0.008	0.102	–	–	0.009	0.083	–	–	0.005	0.058	–	–	0.005	0.047	–	–	0.002	0.031	–	–	0.003	0.029
Model 1 + Model 4	–	–	0.008	0.078	–	–	0.008	0.090	–	–	0.004	0.044	–	–	0.004	0.047	–	–	0.002	0.029	–	–	0.002	0.025
Model 2 + Model 3	–	–	0.012	0.089	–	–	0.010	0.102	–	–	0.008	0.047	–	–	0.006	0.055	–	–	0.003	0.027	–	–	0.004	0.031

(Continues)

TABLE 4 (Continued)

	True Copula: Gumbel + Gaussian																							
	$T = 200$								$T = 500$								$T = 1000$							
	Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian	
Panel 5	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW
Model 2 + Model 4	–	–	0.009	0.083	–	–	0.009	0.091	–	–	0.005	0.047	–	–	0.004	0.051	–	–	0.002	0.028	–	–	0.002	0.027
Model 3 + Model 4	–	–	0.010	0.094	–	–	0.011	0.097	–	–	0.006	0.051	–	–	0.006	0.055	–	–	0.003	0.025	–	–	0.004	0.026
	True Copula: Frank + Gaussian																							
	$T = 200$								$T = 500$								$T = 1000$							
	Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian		Clayton		Gumbel		Frank		Gaussian	
Panel 6	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW	CM	CW
Model 1 + Model 2	–	–	–	–	0.011	0.113	0.009	0.106	–	–	–	–	0.007	0.060	0.005	0.052	–	–	–	–	0.003	0.035	0.003	0.031
Model 1 + Model 3	–	–	–	–	0.010	0.105	0.014	0.085	–	–	–	–	0.006	0.057	0.007	0.049	–	–	–	–	0.003	0.032	0.003	0.026
Model 1 + Model 4	–	–	–	–	0.011	0.102	0.008	0.101	–	–	–	–	0.006	0.056	0.005	0.050	–	–	–	–	0.004	0.031	0.004	0.027
Model 2 + Model 3	–	–	–	–	0.013	0.098	0.013	0.092	–	–	–	–	0.008	0.055	0.006	0.053	–	–	–	–	0.005	0.036	0.003	0.033
Model 2 + Model 4	–	–	–	–	0.009	0.097	0.012	0.089	–	–	–	–	0.004	0.053	0.006	0.045	–	–	–	–	0.002	0.035	0.003	0.028
Model 3 + Model 4	–	–	–	–	0.012	0.102	0.011	0.091	–	–	–	–	0.007	0.053	0.008	0.041	–	–	–	–	0.004	0.029	0.006	0.022

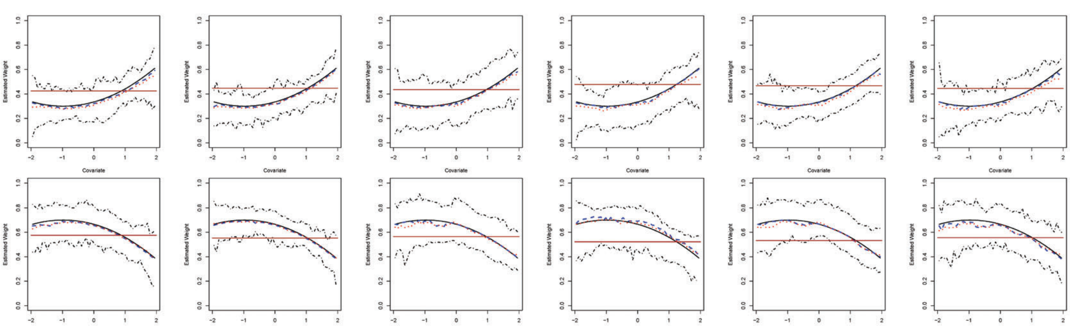
TABLE 5 Mean squared errors (MSEs) of copula parameter estimates and accurate (inaccurate) rates of selection by the proposed conditional mixture copula (CM) and Cai and Wang (2014) constant mixture copula (CW) when the true model is an individual constant copula (Panel 1) and a constant mixture copula (Panel 2)

		True Copula: Clayton							
		$T = 200$							
		Clayton		Gumbel		Frank		Gaussian	
Panel 1		CM	CW	CM	CW	CM	CW	CM	CW
$\theta_{\text{Cl}} = 5$	MSE	0.173	0.121	–	–	–	–	–	–
	Rate	1.000	1.000	(0.000)	(0.000)	(0.000)	(0.000)	(0.013)	(0.000)
$\theta_{\text{Cl}} = 7$	MSE	0.135	0.119	–	–	–	–	–	–
	Rate	1.000	1.000	(0.000)	(0.000)	(0.000)	(0.000)	(0.004)	(0.000)
		$T = 500$							
		Clayton		Gumbel		Frank		Gaussian	
		CM	CW	CM	CW	CM	CW	CM	CW
$\theta_{\text{Cl}} = 5$	MSE	0.104	0.075	–	–	–	–	–	–
	Rate	1.000	1.000	(0.000)	(0.000)	(0.000)	(0.000)	(0.002)	(0.000)
$\theta_{\text{Cl}} = 7$	MSE	0.071	0.062	–	–	–	–	–	–
	Rate	1.000	1.000	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
		$T = 1000$							
		Clayton		Gumbel		Frank		Gaussian	
		CM	CW	CM	CW	CM	CW	CM	CW
$\theta_{\text{Cl}} = 5$	MSE	0.047	0.033	–	–	–	–	–	–
	Rate	1.000	1.000	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
$\theta_{\text{Cl}} = 7$	MSE	0.034	0.029	–	–	–	–	–	–
	Rate	1.000	1.000	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
		True Copula: Clayton + Gumbel (equally weighted)							
		$T = 200$							
		Clayton		Gumbel		Frank		Gaussian	
Panel 2		CM	CW	CM	CW	CM	CW	CM	CW
$\theta_{\text{Cl}} = 5, \theta_{\text{Gu}} = 4$	MSE	0.253	0.194	0.367	0.231	–	–	–	–
	Rate	1.000	1.000	0.859	0.831	(0.118)	(0.125)	(0.236)	(0.221)
$\theta_{\text{Cl}} = 7, \theta_{\text{Gu}} = 6$	MSE	0.385	0.253	0.513	0.277	–	–	–	–
	Rate	1.000	1.000	0.893	0.865	(0.127)	(0.167)	(0.262)	(0.258)
		$T = 500$							
		Clayton		Gumbel		Frank		Gaussian	
		CM	CW	CM	CW	CM	CW	CM	CW
$\theta_{\text{Cl}} = 5, \theta_{\text{Gu}} = 4$	MSE	0.172	0.110	0.218	0.138	–	–	–	–
	Rate	1.000	1.000	0.941	0.922	(0.053)	(0.051)	(0.115)	(0.107)

(Continues)

TABLE 5 (Continued)

		True Copula: Clayton + Gumbel (equally weighted)							
		T = 200							
		Clayton		Gumbel		Frank		Gaussian	
Panel 2		CM	CW	CM	CW	CM	CW	CM	CW
$\theta_{\text{Cl}} = 7, \theta_{\text{Gu}} = 6$	MSE	0.243	0.183	0.324	0.206	–	–	–	–
	Rate	1.000	1.000	0.949	0.958	(0.044)	(0.049)	(0.127)	(0.139)
		T = 1000							
		Clayton		Gumbel		Frank		Gaussian	
		CM	CW	CM	CW	CM	CW	CM	CW
$\theta_{\text{Cl}} = 5, \theta_{\text{Gu}} = 4$	MSE	0.106	0.071	0.133	0.079	–	–	–	–
	Rate	1.000	1.000	0.981	0.991	(0.009)	(0.006)	(0.035)	(0.022)
$\theta_{\text{Cl}} = 7, \theta_{\text{Gu}} = 6$	MSE	0.166	0.105	0.218	0.133	–	–	–	–
	Rate	1.000	1.000	0.983	0.977	(0.005)	(0.002)	(0.030)	(0.031)



(a) ω_{Cl} with $\theta_{Cl,\text{Mod 1}}$ & ω_{Fr} with $\theta_{Fr,\text{Mod 2}}$ (b) ω_{Cl} with $\theta_{Cl,\text{Mod 1}}$ & ω_{Fr} with $\theta_{Fr,\text{Mod 3}}$ (c) ω_{Cl} with $\theta_{Cl,\text{Mod 1}}$ & ω_{Fr} with $\theta_{Fr,\text{Mod 4}}$ (d) ω_{Cl} with $\theta_{Cl,\text{Mod 2}}$ & ω_{Fr} with $\theta_{Fr,\text{Mod 3}}$ (e) ω_{Cl} with $\theta_{Cl,\text{Mod 2}}$ & ω_{Fr} with $\theta_{Fr,\text{Mod 4}}$ (f) ω_{Cl} with $\theta_{Cl,\text{Mod 3}}$ & ω_{Fr} with $\theta_{Fr,\text{Mod 4}}$

FIGURE 3 Estimated paths for weights ($\boldsymbol{\omega}$) when the true model is a combination of the Clayton and Frank copulas. *Notes*. Panel (a) displays the weight estimates for the Clayton (up) and Frank (down) copulas when θ_{Cl} = Model 1 and θ_{Fr} = Model 2. Panel (b) displays the weight estimates for the Clayton (up) and Frank (down) copulas when θ_{Cl} = Model 1 and θ_{Fr} = Model 3. Panel (c) displays the weight estimates for the Clayton (up) and Frank (down) copulas when θ_{Cl} = Model 1 and θ_{Fr} = Model 4. Panel (d) displays the weight estimates for the Clayton (up) and Frank (down) copulas when θ_{Cl} = Model 2 and θ_{Fr} = Model 3. Panel (e) displays the weight estimates for the Clayton (up) and Frank (down) copulas when θ_{Cl} = Model 2 and θ_{Fr} = Model 4. Panel (f) displays the weight estimates for the Clayton (up) and Frank (down) copulas when θ_{Cl} = Model 3 and θ_{Fr} = Model 4. The black solid line denotes the true path of $\omega(z)$. The red dotted line and the blue dashed line, respectively, denote the mean and median of the weight estimates at the grid points with 1000 simulations. The brown solid line denotes the mean of the estimates with 1000 simulations by Hu (2006), and Cai and Wang (2014). The black dotted-dashed lines denote the 5% and 95% percentiles of the weight estimates at the grid points. The sample size $T = 1000$ in all panels

4 | AN EMPIRICAL ILLUSTRATION

In this section we apply the proposed methods to investigate how the dependence structures among the international equity markets evolve with the volatility in exchange rate markets. Equity price and exchange rate are two important financial variables that are closely linked to each other. Shocks in the exchange rate market cause fluctuations in the value of a domestic currency, impacting trade flows, capital movements and equity prices. Therefore, understanding the relation between exchange rate markets and equity markets and the spillover effect of exchange rate markets on equity prices has substantive implications in terms of risk management.

From CRSP, we collect the weekly MSCI equity prices in four developed economies (France, Germany, the United States, and the United Kingdom) and the weekly exchange rates among the U.S. dollar (USD), the British pound (GBP) and the euro (EUR). The observations are between July 1, 1999 and July 11, 2018. We transform the weekly equity prices and exchange rates into log returns by taking the first-order differences on their logarithmic levels. The first panel in Table 6 documents some summary statistics of the weekly log returns of the equity prices and exchange rates. One can observe that the European stock markets exhibit larger fluctuations than the United States market during the sample period, while the latter gives relatively higher average returns. Compared with the equity markets, both returns and fluctuations are lower in these exchange rate markets. The Jarque–Bera test results show that the null hypothesis of normality is rejected for all six return series. The second panel in Table 6 displays that the linear (Pearson) correlation coefficients across the four equity markets are very high, which is expected considering the economic synchronization of the four developed economies.

Preliminary examinations suggest that the autocorrelation and conditional heteroscedasticity exist in these log return series. Thus, we follow Zimmer (2012) and use an AR-GARCH process to model the conditional mean and conditional variance. Specifically, we fit the series of returns to an AR(1)-GARCH(1,1) process written as

$$x_{it} = \gamma_{i0} + \gamma_{i1} x_{i,t-1} + e_{it}, \quad e_{it} = \sigma_{it} \epsilon_{it},$$

where x_{it} denotes the return at time t for country i. The innovations ϵ_{it} are assumed to be i.i.d. The conditional variance is defined as

$$\sigma_{it}^2 = \alpha_{i0} + \alpha_{i1} \sigma_{i,t-1}^2 + \alpha_{i2} e_{i,t-1}^2,$$

where α_{i0}, α_{i1}, and α_{i2} are parameters of GARCH(1,1) for country i with $\alpha_{i0} > 0$, $\alpha_{i1} \geq 0$, $\alpha_{i2} \geq 0$ and $\alpha_{i1} + \alpha_{i2} < 1$. Table 7 summarizes the coefficients of the AR(1)-GARCH(1,1) filtering and shows that most estimates are statistically significant. For all cases, the Ljung–Box test statistics are not significant at any conventional levels, implying the effectiveness of the AR-GARCH procedure in filtering out the linear dependence in the series of returns.

Because our target is to investigate the dynamic pattern of the dependence among the international equity markets along the path of the exchange rate volatility, we first, respectively, estimate the volatility of USD-EUR and USD-GBP by the AR(1)-GARCH(1,1) model discussed above. Figure 4 demonstrates the time series plots of the four countries' weekly equity prices along the estimated volatility of the exchange rates (black dashed) for USD-GBP and USD-EUR. Unlike the equity prices which substantially fluctuate during the sample period, the volatility paths of exchange rates are relatively stable. As a matter of fact, the exchange rate markets witnessed a tranquil period during 1999–2007 with the estimated volatility of USD-EUR and USD-GBP

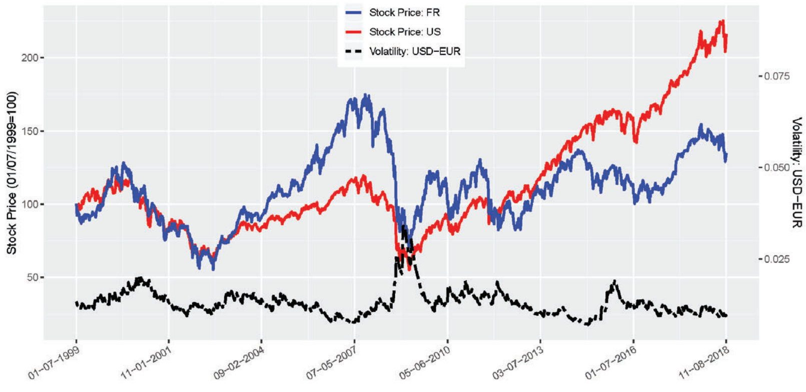
TABLE 6 Summary statistics

	FR	DE	US	UK	USD-EUR	USD-GBP
Panel 1: Summary statistics						
Mean (%)	0.037	0.031	0.078	−0.007	0.003	0.023
Median (%)	0.266	0.282	0.244	0.155	0.039	−0.038
Min (%)	−17.581	−17.504	−16.748	−15.220	−9.010	−5.547
Max (%)	12.829	13.977	10.344	10.915	6.085	10.222
Std. Dev	0.033	0.035	0.023	0.028	0.013	0.013
Skewness	−0.514	−0.698	−0.678	−0.521	−0.256	0.704
Kurtosis	2.916	3.081	5.043	3.079	2.788	5.377
JB	415^{***}	497^{***}	1184^{***}	459^{***}	349^{***}	1341^{***}
Panel 2: Linear correlation coefficients						
	DE	US	UK			
FR	0.932	0.740	0.878			
DE	–	0.740	0.830			
US	–	–	0.735			

Notes. Panel 1 documents the summary statistics of the weekly log returns of the MSCI equity prices in France, Germany, the United States and the United Kingdom, and the weekly log returns of the U.S. dollar-euro and U.S. dollar–British pound exchange rates. *JB* denotes the statistic of the Jarque–Bera test with the null hypothesis of normality. ***indicates rejection of the null at 1%. Panel 2 documents the linear correlation coefficients among the four international equity markets' weekly log returns. The sample period are between July 1, 1999 and July 11, 2018.

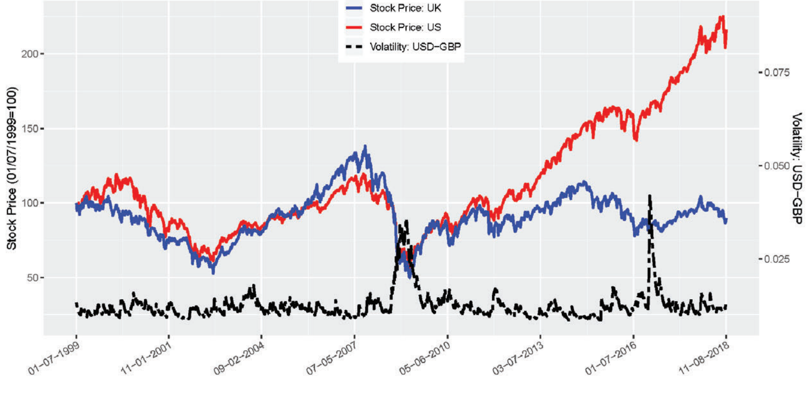
ranged between 0.007 and 0.02, whereas the equity markets in these countries were shocked by a sequence of events such as the recession induced by the burst of the dotcom bubbles, the 9/11 terrorism attack, and two military operations against Afghanistan and Iraq, etc. However, sometimes the tranquility in the exchange rate markets could also be interrupted by a domestic or international event. As can be seen in the first two panels in Figure 4, there is a remarkable peak on the volatility path of USD-EUR in 2008 when the global financial crisis caused severe recession and stock markets crashed in all developed economies. For example, Figure 4(a) shows that just in 2008 the stock price dropped by 46% in France (blue solid) and 40% in the United States (red solid), while the volatility of the USD-EUR exchange rate is nearly tripled at the end of 2008, jumping from 0.012 to 0.034. One could observe similar patterns from the other two pairs. One distinctive feature of the USD-GBP volatility path is that, besides the spike in 2008, there is another remarkable peak in June of 2017 due to panics among investors induced by the passage of the Brexit referendum.

Next, we fit the filtered equity returns (i.e., the residuals) to the conditional mixture copula model that contains the Clayton, Gumbel, and Frank copulas. In the first step, we apply the two proposed model selection procedures to determine which candidate copula(s) should be included in the mixture model. We firstly consider the information criterion method. Because the mixture copula model contains three components, for each pair of markets, we need to respectively estimate $2^3 - 1 = 7$ copula models and calculate their corresponding BIC values. Then, as discussed in Section 2.4, we make a comparison and choose the model with the lowest BIC value among the seven BICs. Table 8 documents the BIC values of the seven models for the three pairs of equity



(a) U.S. - France

(b) U.S. - Germany



(c) U.S. - UK

FIGURE 4 Time series plots for weekly equity prices and estimated volatility of exchange rates. *Notes*. This figure displays four developed economies' weekly MSCI stock prices (July 1, 1999 = 100) and the volatility of exchange rates estimated by the AR(1)-GARCH(1,1) during July 1, 1999 – July 11, 2018. Panel(a) plots the volatility of the exchange rate between the U.S. dollar and the euro (black dashed) and the MSCI stock prices in France (blue solid) and the United States (red solid). Panel (b) plots the volatility of the exchange rate between the US dollar and the euro (black dashed) and the MSCI stock prices in Germany (blue solid) and the United States (red solid). Panel (c) plots the volatility of the exchange rate between the U.S. dollar and the British pound (black dashed) and the MSCI stock prices in the United Kingdom (blue solid) and the United States (red solid)

TABLE 7 The estimates of AR(1)-GARCH(1,1)

	AR(1) Part	GARCH(1, 1) Part			
	γ_1 (*p*-value)	α_0 (*p*-value)	α_1 (*p*-value)	α_2 (*p*-value)	*LB* (*p*-value)
Panel 1: Equity Return					
France	−0.119***	0.000***	0.805***	0.161***	0.755
	(.000)	(.000)	(.000)	(.000)	(.385)
Germany	−0.079**	0.000***	0.836***	0.129***	0.127
	(.018)	(.006)	(.000)	(.000)	(.722)
United States	−0.122***	0.000***	0.795***	0.169***	1.763
	(.000)	(.002)	(.000)	(.000)	(.184)
United Kingdom	−0.088***	0.000***	0.755***	0.181***	1.716
	(.009)	(.000)	(.000)	(.000)	(.190)
Panel 2: Exchange Rate					
USD-EUR	0.043	0.000**	0.898***	0.086***	0.029
	(.176)	(.039)	(.000)	(.000)	(.864)
USD-GBP	0.013	0.000***	0.799***	0.126***	2.160
	(.706)	(.008)	(.000)	(.000)	(.142)

Notes. This table summarizes the results of the AR(1)-GARCH(1,1) filtering. *LB* denotes the statistic of the Ljung-Box test with the null of zero autocorrelation for the residuals filtered by AR(1)-GARCH(1,1). Values in parentheses are corresponding *p*-values. ** and ***, respectively, indicates rejection of the null at 5% and 1%.

markets. It shows that the mixture model with the Clayton and Frank copulas are selected for all three pairs as it exhibits the lowest BIC among the seven BIC values. In other words, based on the comparison among the BICs associated with the seven candidate models, the final mixture copula should be constructed as $\omega_{\text{Cl}}C_{\text{Cl}} + \omega_{\text{Fr}}C_{\text{Fr}}$.

For comparing purposes, we alternatively apply the hypothesis test method to check which component copulas should be kept. Similar to the procedures of the backward elimination, we start with a mixture model with all three component copulas. Then we test whether the three weight parameters ω_{Cl}, ω_{Gu}, and ω_{Fr}, respectively, equal to zero, and remove the copula whose weight parameter's *p*-value is the highest among those greater than 0.05, the significance level. We then refit the model until the *p*-values of the weight parameters of the remaining component copulas are all lower than 0.05. The sequence of the hypothesis tests and the results are displayed in Table 9. In Panel 1, the hypothesis test results show that the *p*-values for the Gumbel copula's weight parameters are remarkably greater than 0.05 in all three market pairs, indicating that the weight parameters of the Gumbel copula are insignificantly different from zero and the Gumbel copula should be filtered out from the mixture models in the first step. Next, we refit the mixture model which only contains the Clayton and Frank copulas. Panel 2 of Table 9 suggests that both component copulas should be kept as the *p*-values of their weight parameters are lower than 0.05 in all three market pairs. That is, the final mixture copula selected by the hypothesis test method is consistent with the one selected by the information criterion method.

TABLE 8 BIC values for different copula models

	United States-Germany	United States-France	United States-United Kingdom
Clayton	−11,800.282	−11,508.044	−11,789.353
Gumbel	−11,043.538	−10,738.931	−11,056.510
Frank	−10,747.739	−10,763.693	−10,789.578
Clayton+Gumbel	−12,692.052	−12,448.875	−12,705.331
Clayton+Frank	**−12,940.750**	**−12,685.836**	**−12,965.381**
Gumbel+Frank	−12,579.391	−12,356.165	−12,590.426
Clayton+Gumbel+Frank	−11,743.994	−11,542.665	−11,701.498

Notes. This table reports the BIC values of seven individual and mixture copula models for the pairs of United States-Germany, United States-France and United States-United Kingdom. The best models are in bold.

TABLE 9 *p*-values of the hypothesis tests

	U.S.- France	United States–Germany	United States–UK
	***p*-value**	***p*-value**	***p*-value**
Panel 1: Mixture model = $\omega_{\mathrm{Cl}} C_{\mathrm{Cl}} + \omega_{\mathrm{Gu}} C_{\mathrm{Gu}} + \omega_{\mathrm{Fr}} C_{\mathrm{Fr}}$			
$H_0 : \omega_{\mathrm{Cl}} = 0$ v.s. $H_1 : \omega_{\mathrm{Cl}} \neq 0$	.000	.000	.000
$H_0 : \omega_{\mathrm{Gu}} = 0$ v.s. $H_1 : \omega_{\mathrm{Gu}} \neq 0$	.217	.335	.241
$H_0 : \omega_{\mathrm{Fr}} = 0$ v.s. $H_1 : \omega_{\mathrm{Fr}} \neq 0$	.000	.000	.000
Panel 2: Mixture model = $\omega_{\mathrm{Cl}} C_{\mathrm{Cl}} + \omega_{\mathrm{Fr}} C_{\mathrm{Fr}}$			
$H_0 : \omega_{\mathrm{Cl}} = 0$ v.s. $H_1 : \omega_{\mathrm{Cl}} \neq 0$	.000	.000	.000
$H_0 : \omega_{\mathrm{Fr}} = 0$ v.s. $H_1 : \omega_{\mathrm{Fr}} \neq 0$	.000	.000	.000

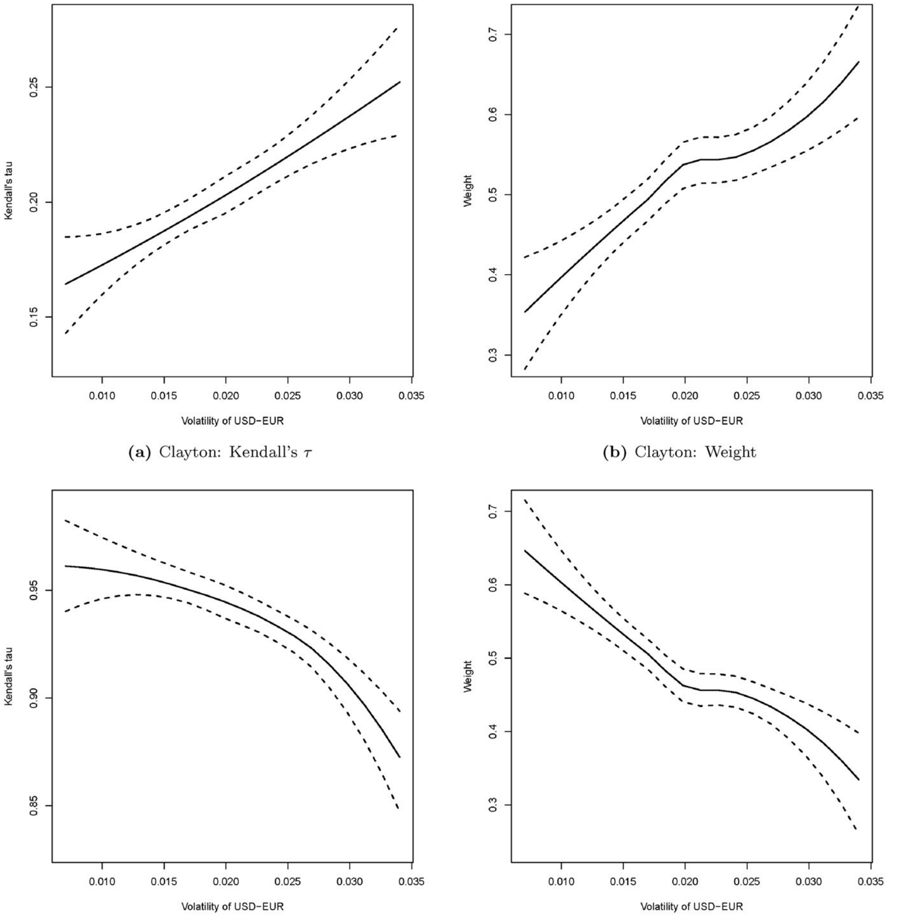
Notes. This table displays the *p*-values for the estimates of the weight parameters in a sequence of hypothesis tests with the .05 significance level. In Panel 1, the mixture model is assumed to contain all three component copulas and three hypothesis tests are respectively implemented. The component copula with the highest *p*-value among those greater than .05 is filtered out. In Panel 2, we refit the mixture model with two component copulas and implement two hypothesis tests. We will exclude the copula whose weight estimate's *p*-value is greater than .05. Otherwise we will keep both component copulas.

When a mixture copula contains many component copulas, it would be efficient if we can simultaneously filter out several candidate copulas in one step. To this end, in this example we also apply the hypothesis test by firstly examining the joint hypothesis tests. Specifically, in the first step, we, respectively, test whether each two of the three copulas' weight parameters are simultaneously equal to zero. Failing to reject the null hypothesis of any of these joint tests indicates that the mixture copula only contains an individual copula. In our example, we reject the null hypotheses of all the joint tests for the three market pairs at the 0.05 significance level, suggesting that the mixture model should include more than one component copula. Next, we implement hypothesis tests to examine whether the three weight parameters are, respectively, equal to zero. The test results show that, for all three pairs, we cannot reject $\omega_{\mathrm{Gu}} = 0$ at any conventional levels. Therefore, the final choice of the mixture model should be constructed by the Clayton and Frank copulas. The detailed results for this sequence of hypothesis tests are available upon request.

Our finding is in line with Acar et al. (2013), Gijbels et al. (2017), and Derumigny and Fermanian (2017) who argue that a combination of the Gaussian and rotated Gumbel copulas outperforms the other models in examining comovements among stock returns along the exchange rate volatility. Given the observed downward comovements among the equity markets when the exchange rates become extremely volatile, it is expected that the Clayton copula should be kept in the mixture model to capture the lower tail dependence. The inclusion of the Frank copula is also intuitive because, as Figure 4 shows, except for the extreme scenario in 2008, in general we do not observe obvious tail dependence of equity returns along the exchange rate volatility over the sample period.

Subsequently, we fit the data to the selected conditional mixture copula model and obtain the estimated weight and copula parameters through the proposed method. Since parameters from different copulas are not directly comparable, here we transform the copulas' parameter estimates into Kendall's τs. As a measure of concordance between random variables, Kendall's τ is invariant to nonlinear transformations and thus can capture nonlinear dependence which is unable to be measured by the linear correlation coefficient. For example, for X_1 and X_2 with their respective CDFs u_1 and u_2, it is possible to express Kenadall's τ in terms of a copula which connects the two random variables as $\tau = 4\int_0^1\int_0^1 C(u_1, u_2; \theta)dC(u_1, u_2; \theta) - 1$. For Clayton, Kendall's $\tau_{\text{Cl}} = \frac{\theta}{\theta+2}$. For Gumbel, $\tau_{\text{Gu}} = 1 - \frac{1}{\theta}$. For Frank, $\tau_{\text{Fr}} = 1 - \frac{4}{\theta}\left(1 - \int_0^\theta \frac{t}{e^t-1}dt\right)$.

Figure 5 demonstrates the estimates (solid curves) of Kendall's τs and the weights of the Clayton and Frank copulas for the U.S.-Germany equity returns along the volatility of USD-EUR, and the 5% and 95% percentiles (dashed curves) obtained through the proposed block bootstrap method. Figure 5(a) shows that the magnitude of the lower tail dependence, measured by Kendall's τ of Clayton, doubles from 0.15 to about 0.3 as the volatility of USD-EUR is more than tripled from about 0.01 to 0.035. Such strengthened asymmetric dependence is further amplified by the increasing weight associated with the Clayton copula, as displayed by Figure 5(b): as the exchange rate between the U.S. dollar and the euro becomes increasingly volatile, the effect of the lower tail dependence turns to be more dominant. On the contrary, both the estimates of Kendall's τ and the weight of the Frank copula decrease as the volatility of USD-EUR increases. Even though the magnitude of the decline in Kendall's τ for the Frank copula is relatively small, its dominance in the dependence structure is remarkably weakened due to the decreased weight in the mixture model. For the pairs of United States–France and United States-United Kingdom, Figures 6 and 7 show quite similar patterns: when the exchange rate becomes increasingly volatile, the lower tail dependence is strengthened as the estimates of Kendall's τ and the weight associated with the Clayton copula simultaneously increase, while both Kendall's τ and the weight of the Frank copula shrink. Given the close economic and political connections of the three countries to the United States, the similar patterns of Kendall's τs along the exchange rate volatility should be expected. When the exchange rate becomes extremely volatile (e.g., during the global economic recession), the weight of the Clayton copula exceeds that of the Frank copula, indicating that the lower tail dependence dominates the mixture dependence structure so that the two equity markets exhibit a higher probability to crash simultaneously. The fact that higher volatility in exchange rate markets is associated with more extreme asymmetric dependence among equity markets is not only in line with our observations in Figures 5–7, but also consistent with the findings in the literature such as Cai and Wang (2014). An extreme jump in the exchange rate usually leads to an extreme comovement in equity markets. As Cai and Wang (2014) argue, when a sudden and unexpected shock hits an economy with a very active currency market, transmission through the latter makes a downside comovement of equity markets more likely than in a calm period of the exchange rate



(a) Clayton: Kendall's τ (b) Clayton: Weight

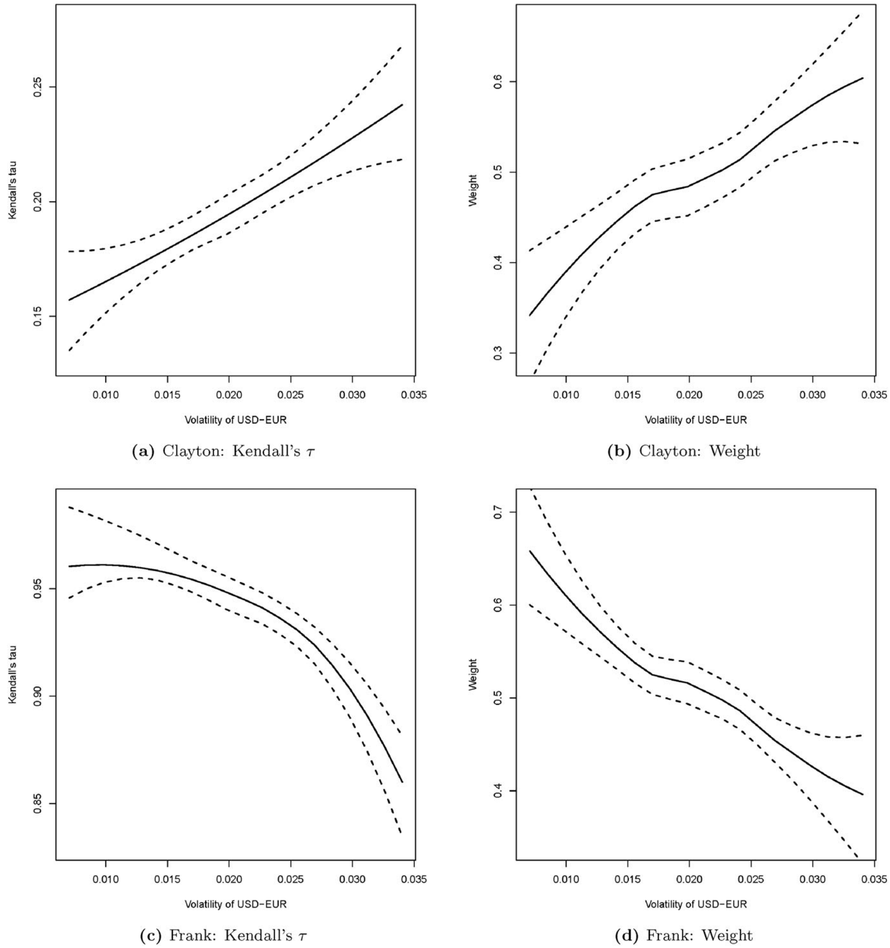
(c) Frank: Kendall's τ (d) Frank: Weight

FIGURE 5 United States–Germany: The estimated paths for Kenall's $\boldsymbol{\tau}$s and weights. *Notes*. This figure displays the estimates of Kendall's τs and the weights of the Clayton and Frank copulas for equity returns between United States and Germany, along the estimated volatility of USD-EUR. Panel (a) shows the Kendall's τ estimate of Clayton along the estimated volatility of USD-EUR. Panel (b) shows the weight estimate of Clayton along the estimated volatility of USD-EUR. Panel (c) shows the Kendall's τ estimate of Frank along the estimated volatility of USD-EUR. Panel (d) shows the weight estimate of Frank along the estimated volatility of USD-EUR. The two dashed lines in all four panels denote the 5% and 95% percentiles. The data are at weekly frequency and span from July 1, 1999 to July 11, 2018

market. This may partially explain why the Clayton copula's weight and Kendall's τ both increase as the exchange rate market becomes more volatile.

5 | CONCLUSION

This paper proposes a semi-parametric conditional mixture copula model in which both weight and copula parameters can vary with a covariate in a nonparametric way. The conditional mixture copula exploits the advantages of both the conditional copula which can capture a covariate's



(a) Clayton: Kendall's τ

(b) Clayton: Weight

(c) Frank: Kendall's τ

(d) Frank: Weight

FIGURE 6 United States–France: The estimated paths for Kenall's $\boldsymbol{\tau}$s and weights. *Notes*. This figure displays the estimates of Kendall's τs and the weights of the Clayton and Frank copulas for equity returns between United States and France, along the estimated volatility of USD-EUR. Panel (a) shows the Kendall's τ estimate of Clayton along the estimated volatility of USD-EUR. Panel (b) shows the weight estimate of Clayton along the estimated volatility of USD-EUR. Panel (c) shows the Kendall's τ estimate of Frank along the estimated volatility of USD-EUR. Panel (d) shows the weight estimate of Frank along the estimated volatility of USD-EUR. The two dashed lines in all four panels denote the 5% and 95% percentiles. The data are at weekly frequency and span from July 1, 1999 to July 11, 2018

impact on the degree of dependence (see Cai & Wang, 2014), and the mixture copula which can combine copula families with different dependence patterns (see Cai & Wang's, 2014). Therefore, it provides extra flexibility and an unified way for practitioners to measure the dependence pattern and the degree of dependence.

In the theoretical part, we provide a two-step estimation procedure to separately estimate the marginal distributions and the weight and copula parameters in the model, and the large sample properties of these estimators are derived. Moreover, we introduce two model selection approaches to choose an appropriate conditional mixture copula model from a large copula candidate set. Monte Carlo simulation results confirm that the proposed estimation and model

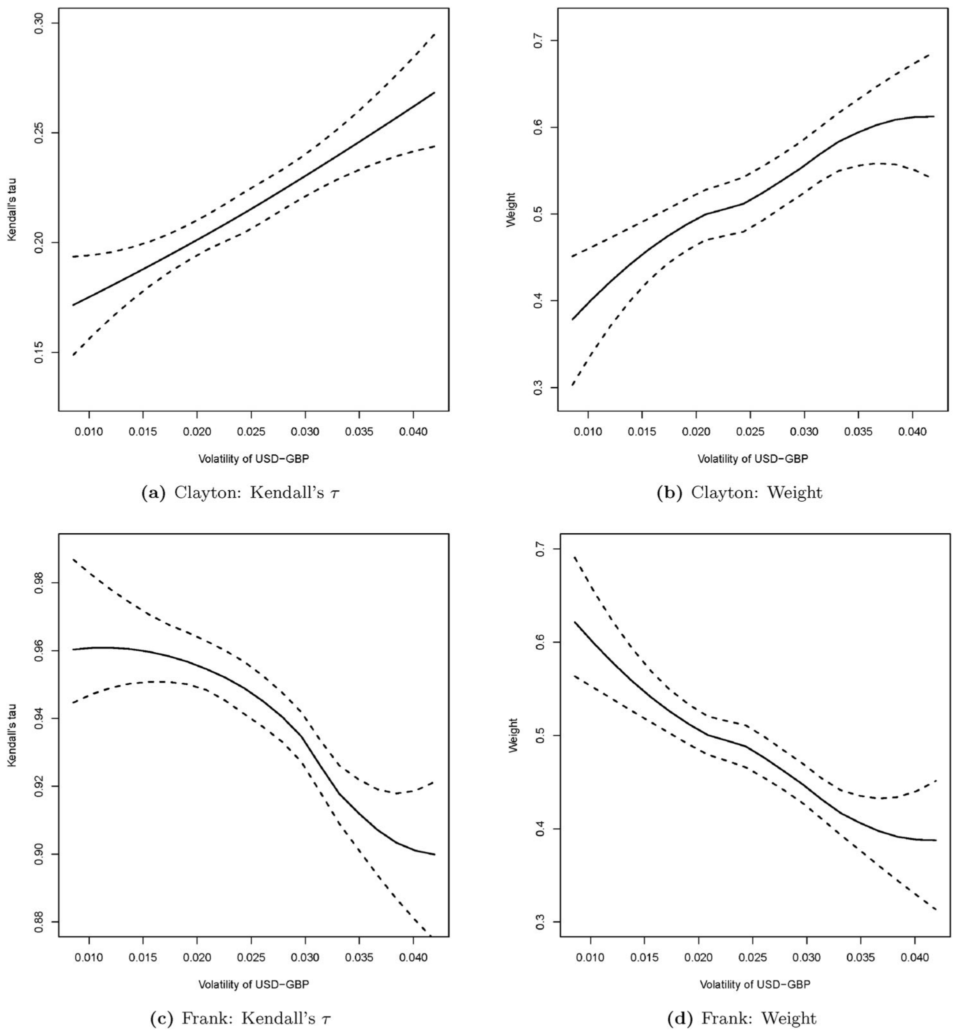


FIGURE 7 United States–United Kingdom: The estimated paths for Kenall's $\boldsymbol{\tau}$s and weights. *Notes*. This figure displays the estimates of Kendall's τs and the weights of the Clayton and Frank copulas for equity returns between United States and United Kingdom, along the estimated volatility of USD–GBP. Panel (a) shows the Kendall's τ estimate of Clayton along the estimated volatility of USD-GBP. Panel (b) shows the weight estimate of Clayton along the estimated volatility of USD-GBP. Panel (c) shows the Kendall's τ estimate of Frank along the estimated volatility of USD-GBP. Panel (d) shows the weight estimate of Frank along the estimated volatility of USD-GBP. The two dashed lines in all four panels denote the 5% and 95% percentiles. The data are at weekly frequency and span from July 1, 1999 to July 11, 2018

selection procedures perform reasonably well in estimating unknown parameters and selecting component copulas. The only exception is when weights and copula parameters in a mixture copula are constants: simulation results show that although the proposed conditional mixture copula estimation method still provides accurate copula selection, its estimation accuracy, measured by MSE, becomes slightly lower than the constant mixture copula estimation method proposed by Cai and Wang's (2014). In an empirical illustration, we apply the proposed methods to examine how the dependence structures among the international equity markets evolve with the volatility in the exchange rate markets and find that both the weight of the Clayton copula and the degree of the lower tail dependence among the equity markets remarkably increase when the exchange rate markets become more volatile.

In practice, because whether weights and copula parameters are constants or not is unknown ex ante, the conditional mixture copula is an ideal model if practitioners have strong belief that both the pattern of dependence (summarized by weights of component copulas) and degree of dependence (measured by copula parameters) are affected by certain covariate. For example, applying the proposed conditional mixture copula, practitioners can extend Cai and Wang's (2014) and investigate how strength and direction of comovement among housing markets in the United States evolved with certain economic indicator such as per capita disposable income in the past four decades. In this analysis, as in the empirical illustration in Section 4, we need the conditional mixture copula to detect effects of the covariate on dependence structure. Alternatively, if practitioners need a quick examination on tail dependence and degree of dependence, the CW method would be an useful model with fewer parameters to estimate. In this study, we skip testing the irrelevance of the covariate and refer interested readers to Cai and Wang's (2014) for discussions and empirical illustrations in the i.i.d. scenario. Testing the "simplifying assumption" with weakly dependent data deserves a separate study in the future.

Some interesting future research topics related to this article should be mentioned. First, the proposed method can be extended to a higher dimension of Z because the dependence structure could be potentially affected by several economic variables simultaneously. In other words, the copula parameter $\theta_k(\cdot)$ in Equation (1) can be written as $\theta_k(\gamma^\intercal \mathbf{Z}_t)$, where $\theta_k(\gamma^\intercal \mathbf{Z}_t)$ now is a flexible function of the so-called single-index $\gamma^\intercal \mathbf{Z}_t$, that is, a linear combination of p_z-dimensional economics variable $\mathbf{Z}_t$ with γ being a p_z-dimensional vector loading, and $\mathbf{Z}_t = (Z_{1t}, \dots, Z_{p_z t})^\intercal$. Second, our model can be applied by empirical practitioners to study how dependence structures among the international stock markets abruptly changed amid the outbreak of the COVID-19 pandemic when equity prices crashed in global markets.

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REFERENCES

Abegaz, F., Gijbels, I., & Veraverbeke, N. (2012). Semiparametric estimation of conditional copulas. *Journal of Multivariate Analysis*, *110*, 43–73.

Acar, E., Craiu, R., & Yao, F. (2011). Dependence calibration in conditional copulas: A nonparametric approach. *Biometrics*, *67*, 445–453.

Acar, E., Craiu, R., & Yao, F. (2013). Statistical testing of covariate effects in conditional copula models. *Electronic Journal of Statistics*, *7*, 2822–2850.

Cai, Z. (2002). Regression quantiles for time series. *Econometric Theory*, *18*, 169–192.

Cai, Z., & Wang, X. (2014). Selection of mixed copula model via penalized likelihood. *Journal of the American Statistical Association*, *109*, 788–801.

Chen, X., & Fan, Y. (2006a). Estimation of copula-based semiparametric time series models. *Journal of Econometrics*, *130*, 307–335.

Chen, X., & Fan, Y. (2006b). Estimation and model selection of semiparametric copula-based multivariate dynamic models under copula misspecification. *Journal of Econometrics*, *135*, 125–154.

Chollete, L., Peña, V., & Lu, C. (2005). Comovement of international financial markets. *Unpublished Manuscript*.

Derumigny, A., & Fermanian, J. (2017). About tests of the "simplifying" assumption for conditional copulas. *Dependence Modeling*, *5*, 154–197.

Fan, J., & Gijbels, I. (1996). *Local polynomial modelling and its applications* (Vol. *66*). Chapman & Hall.

Fan, J., Zhang, C., & Zhang, J. (2001). Generalized likelihood ratio statistics and wilks phenomenon. *Annals of Statistics*, *29*, 153–193.

Fan, Y., & Patton, A. J. (2014). Copulas in econometrics. *Annual Review of Economics*, *6*, 179–200.

Fermanian, J., & Lopez, O. (2018). Single-index copulas. *Journal of Multivariate Analysis*, *165*, 27–55.

Garcia, R., & Tsafack, G. (2011). Dependence structure and extreme comovements in international equity and bond markets. *Journal of Banking & Finance*, *35*, 1954–1970.

Giacomini, E., Hardle, W., & Spokoiny, V. (2009). Inhomogeneous dependency modelling with time varying copulae. *Journal of Business and Economic Statistics*, *27*, 224–234.

Gijbels, I., Omelka, M., & Veraverbeke, N. (2017). Nonparametric testing for no covariate effects in conditional copulas. *Statistics*, *51*, 475–509.

Hafner, C., & Manner, H. (2012). Dynamic stochastic copula models: Estimation, inference and applications. *Journal of Applied Econometrics*, *27*, 269–295.

Hu, L. (2006). Dependence patterns across financial markets: A mixed copula approach. *Applied Financial Economics*, *16*, 717–729.

Huang, M., Li, R., & Wang, S. (2013). Nonparametric mixture of regression models. *Journal of the American Statistical Association*, *108*, 929–941.

Liu, G., Long, W., Zhang, X., & Li, Q. (2019). Detecting financial data dependence structure by averaging mixture copulas. *Econometric Theory*, *35*, 777–815.

Longin, F., and Solnik, B. (2001). Extreme Correlation of International Equity Markets. *Journal of Finance*, *56*, 649–676.

Patton, A. (2006). Modeling asymmetric exchange rate dependence. *International Economic Review*, *47*, 527–556.

Patton, A. (2012). A review of copula models for economic time series. *Journal of Multivariate Analysis*, *110*, 4–18.

Sklar, A. (1959). Fonctions de répartition à n dimensions et leurs marges. *Publications de l'Institut de Statistique de L'Université de Paris*, *8*, 229–231.

Zimmer, D. (2012). The role of copulas in the housing crisis. *Review of Economics and Statistics*, *94*, 607–620.

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Additional supporting information may be found online in the Supporting Information section at the end of this article.

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