The Inverse of a Partitioned Matrix

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Consider a pair A, B of $n \times n$ matrices, partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where A_{11} and B_{11} are $k \times k$ matrices. Suppose that A is nonsingular and $B = A^{-1}$. In this note it will be shown how to derive the B_{ij} 's in terms of the A_{ij} 's, given that

$$\det(A_{11}) \neq 0 \text{ and } \det(A_{22}) \neq 0. \tag{1}$$

If $B = A^{-1}$, then,

$$\begin{aligned} AB &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \qquad (2) \\ &= \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix} \\ &= \begin{pmatrix} I_k & O_{k,n-k} \\ O_{n-k,k} & I_{n-k} \end{pmatrix}, \end{aligned}$$

where as usual I denotes the unit matrix and O a zero matrix, with sizes indicated by the subscripts involved. To solve (2), we need to solve four matrix equations:

$$A_{11}B_{11} + A_{12}B_{21} = I_k, \tag{3}$$
$$A_{11}B_{12} + A_{12}B_{22} = O_{k,n-k}, \tag{4}$$
$$A_{21}B_{11} + A_{22}B_{21} = O_{n-k,k} \tag{5}$$
$$A_{21}B_{12} + A_{22}B_{22} = I_{n-k} \tag{6}$$

It follows from (4) and (5) that

$$B_{12} = -A_{11}^{-1}A_{12}B_{22}, \tag{7}$$
$$B_{21} = -A_{22}^{-1}A_{21}B_{11}, \tag{8}$$

so that (3) and (6) become

$$\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)B_{11} = I_k,$$
$$\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)B_{22} = I_{n-k}.$$

Hence,

$$B_{11} = \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1}$$
$$B_{22} = \left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1}.$$

Substituting these solutions in (7) and (8) it follows that

$$B_{12} = -A_{11}^{-1}A_{12}\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1}$$
$$B_{21} = -A_{22}^{-1}A_{21}\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1}.$$

Thus,

$$A^{-1} = \begin{pmatrix} \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} & -A_{11}^{-1}A_{12}\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1} \\ -A_{22}^{-1}A_{21}\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} & \left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1} \end{pmatrix}.$$

Moreover, since $AA^{-1} = I_n$ implies $A^{-1}A = I_n$, we also have

$$A^{-1} = \begin{pmatrix} \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} & -\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1}A_{12}A_{22}^{-1} \\ -\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1}A_{21}A_{11}^{-1} & \left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1} \end{pmatrix}.$$