(b) From the spectral decomposition

$$A = \Gamma\Lambda\Gamma',$$

we obtain

$$\text{rank}(A) = \text{rank}(\Lambda) = \text{tr}(\Lambda) = r,$$

where r is the number of characteristic roots with value 1.

(c) Let $\text{rank}(A) = \text{rank}(\Lambda) = n$, then $\Lambda = I_n$ and

$$A = \Gamma\Lambda\Gamma' = I_n .$$

(a)–(c) follow from the definition of an idempotent matrix.

A.12 Generalized Inverse

Definition A.62 *Let A be an $m \times n$-matrix. Then a matrix $A^- : n \times m$ is said to be a generalized inverse of A if*

$$AA^-A = A$$

holds (see Rao (1973a, p. 24).

Theorem A.63 *A generalized inverse always exists although it is not unique in general.*

Proof: Assume $\text{rank}(A) = r$. According to the singular-value decomposition (Theorem A.32), we have

$$\underset{m,n}{A} = \underset{m,r}{U}\,\underset{r,r}{L}\,\underset{r,n}{V'}$$

with $U'U = I_r$ and $V'V = I_r$ and

$$L = \text{diag}(l_1, \cdots, l_r), \quad l_i > 0.$$

Then

$$A^- = V \begin{pmatrix} L^{-1} & X \\ Y & Z \end{pmatrix} U'$$

(X, Y and Z are arbitrary matrices of suitable dimensions) is a g-inverse of A. Using Theorem A.33, namely,

$$A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

with X nonsingular, we have

$$A^- = \begin{pmatrix} X^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

as a special g-inverse.

Definition A.64 (Moore-Penrose inverse) *A matrix A^+ satisfying the following conditions is called the Moore-Penrose inverse of A:*

(i) $AA^+A = A$,

(ii) $A^+AA^+ = A^+$,

(iii) $(A^+A)' = A^+A$,

(iv) $(AA^+)' = AA^+$.

A^+ is unique.

Theorem A.65 *For any matrix $A : m \times n$ and any g-inverse $A^- : m \times n$, we have*

(i) *A^-A and AA^- are idempotent.*

(ii) $\text{rank}(A) = \text{rank}(AA^-) = \text{rank}(A^-A)$.

(iii) $\text{rank}(A) \le \text{rank}(A^-)$.

Proof:

(a) Using the definition of g-inverse,

$$(A^-A)(A^-A) = A^-(AA^-A) = A^-A.$$

(b) According to Theorem A.23 (iv), we get

$$\text{rank}(A) = \text{rank}(AA^-A) \le \text{rank}(A^-A) \le \text{rank}(A),$$

that is, $\text{rank}(A^-A) = \text{rank}(A)$. Analogously, we see that $\text{rank}(A) = \text{rank}(AA^-)$.

(c) $\text{rank}(A) = \text{rank}(AA^-A) \le \text{rank}(AA^-) \le \text{rank}(A^-)$.

Theorem A.66 *Let A be an $m \times n$-matrix. Then*

(i) *A regular $\Rightarrow A^+ = A^{-1}$.*

(ii) $(A^+)^+ = A$.

(iii) $(A^+)' = (A')^+$.

(iv) $\text{rank}(A) = \text{rank}(A^+) = \text{rank}(A^+A) = \text{rank}(AA^+)$.

(v) *A an orthogonal projector $\Rightarrow A^+ = A$.*

(vi) $\text{rank}(A) : m \times n = m \Rightarrow A^+ = A'(AA')^{-1}$ *and* $AA^+ = I_m$.

(vii) $\text{rank}(A) : m \times n = n \Rightarrow A^+ = (A'A)^{-1}A'$ *and* $A^+A = I_n$.

(viii) *If $P : m \times m$ and $Q : n \times n$ are orthogonal $\Rightarrow$ $(PAQ)^+ = Q^{-1}A^+P^{-1}$.*

(ix) $(A'A)^+ = A^+(A')^+$ *and* $(AA')^+ = (A')^+A^+$.

(x) $A^+ = (A'A)^+A' = A'(AA')^+$.

For further details see Rao and Mitra (1971).

Theorem A.67 (Baksalary, Kala and Klaczynski (1983)) *Let $M : n \times n \geq 0$ and $N : m \times n$ be any matrices. Then*

$$M - N'(NM^+N')^+N \geq 0$$

if and only if

$$\mathcal{R}(N'NM) \subset \mathcal{R}(M).$$

Theorem A.68 *Let A be any square $n \times n$-matrix and a be an n-vector with $a \notin \mathcal{R}(A)$. Then a g-inverse of $A + aa'$ is given by*

$$\begin{aligned}(A + aa')^- &= A^- - \frac{A^-aa'U'U}{a'U'Ua} \\ &\quad - \frac{VV'aa'A^-}{a'VV'a} + \phi\frac{VV'aa'U'U}{(a'U'Ua)(a'VV'a)},\end{aligned}$$

with A^- any g-inverse of A and

$$\phi = 1 + a'A^-a, \quad U = I - AA^-, \quad V = I - A^-A.$$

Proof: Straightforward by checking $AA^-A = A$.

Theorem A.69 *Let A be a square $n \times n$-matrix. Then we have the following results:*

(i) *Assume a, b are vectors with $a, b \in \mathcal{R}(A)$, and let A be symmetric. Then the bilinear form $a'A^-b$ is invariant to the choice of A^-.*

(ii) *$A(A'A)^-A'$ is invariant to the choice of $(A'A)^-$.*

Proof:

(a) $a, b \in \mathcal{R}(A) \Rightarrow a = Ac$ and $b = Ad$. Using the symmetry of A gives

$$\begin{aligned}a'A^-b &= c'A'A^-Ad \\ &= c'Ad.\end{aligned}$$

(b) Using the rowwise representation of A as $A = \begin{pmatrix} a_1' \\ \vdots \\ a_n' \end{pmatrix}$ gives

$$A(A'A)^-A' = (a_i'(A'A)^-a_j).$$

Since $A'A$ is symmetric, we may conclude then (i) that all bilinear forms $a_i'(A'A)a_j$ are invariant to the choice of $(A'A)^-$, and hence (ii) is proved.

Theorem A.70 *Let $A : n \times n$ be symmetric, $a \in \mathcal{R}(A)$, $b \in \mathcal{R}(A)$, and assume $1 + b'A^+a \neq 0$. Then*

$$(A + ab')^+ = A^+ - \frac{A^+ab'A^+}{1 + b'A^+a}.$$

Proof: Straightforward, using Theorems A.68 and A.69.

Theorem A.71 *Let $A : n \times n$ be symmetric, a be an n-vector, and $\alpha > 0$ be any scalar. Then the following statements are equivalent:*

(i) $\alpha A - aa' \geq 0$.

(ii) $A \geq 0$, $a \in \mathcal{R}(A)$, *and* $a'A^-a \leq \alpha$, *with* A^- *being any g-inverse of* A.

Proof:

(i) $\Rightarrow$ (ii): $\alpha A - aa' \geq 0 \Rightarrow \alpha A = (\alpha A - aa') + aa' \geq 0 \Rightarrow A \geq 0$. Using Theorem A.31 for $\alpha A - aa' \geq 0$, we have $\alpha A - aa' = BB$, and, hence,

$$\begin{aligned} & \alpha A = BB + aa' = (B, a)(B, a)'. \\ \Rightarrow \quad & \mathcal{R}(\alpha A) = \mathcal{R}(A) = \mathcal{R}(B, a) \\ \Rightarrow \quad & a \in \mathcal{R}(A) \\ \Rightarrow \quad & a = Ac \quad \text{with} \quad c \in \mathbb{R}^n \\ \Rightarrow \quad & a'A^-a = c'Ac. \end{aligned}$$

As $\alpha A - aa' \geq 0 \quad \Rightarrow$

$$x'(\alpha A - aa')x \geq 0$$

for any vector x, choosing $x = c$, we have

$$\begin{aligned} \alpha c'Ac - c'aa'c &= \alpha c'Ac - (c'Ac)^2 \geq 0, \\ &\Rightarrow c'Ac \leq \alpha. \end{aligned}$$

(ii) $\Rightarrow$ (i): Let $x \in \mathbb{R}^n$ be any vector. Then, using Theorem A.54,

$$\begin{aligned} x'(\alpha A - aa')x &= \alpha x'Ax - (x'a)^2 \\ &= \alpha x'Ax - (x'Ac)^2 \\ &\geq \alpha x'Ax - (x'Ax)(c'Ac) \end{aligned}$$

$$\Rightarrow \quad x'(\alpha A - aa')x \geq (x'Ax)(\alpha - c'Ac).$$

In (ii) we have assumed $A \geq 0$ and $c'Ac = a'A^-a \leq \alpha$. Hence, $\alpha A - aa' \geq 0$.

Note: This theorem is due to Baksalary and Kala (1983). The version given here and the proof are formulated by G. Trenkler.

Theorem A.72 *For any matrix A we have*

$$A'A = 0 \quad \textit{if and only if } A = 0.$$

Proof:

(a) $A = 0 \Rightarrow A'A = 0.$

(b) Let $A'A = 0$, and let $A = (a_{(1)}, \cdots, a_{(n)})$ be the columnwise presentation. Then

$$A'A = (a'_{(i)} a_{(j)}) = 0,$$

so that all the elements on the diagonal are zero: $a'_{(i)} a_{(i)} = 0 \Rightarrow a_{(i)} = 0$ and $A = 0$.

Theorem A.73 *Let $X \neq 0$ be an $m \times n$-matrix and A an $n \times n$-matrix. Then*

$$X'XAX'X = X'X \quad \Rightarrow \quad XAX'X = X \quad \textit{and} \quad X'XAX' = X'.$$

Proof: As $X \neq 0$ and $X'X \neq 0$, we have

$$\begin{aligned} X'XAX'X - X'X = (X'XA - I)X'X &= 0 \\ \Rightarrow (X'XA - I) &= 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow 0 &= (X'XA - I)(X'XAX'X - X'X) \\ &= (X'XAX' - X')(XAX'X - X) = Y'Y, \end{aligned}$$

so that (by Theorem A.72) $Y = 0$, and, hence $XAX'X = X$.

Corollary: Let $X \neq 0$ be an $m \times n$-matrix and A and b $n \times n$-matrices. Then

$$AX'X = BX'X \quad \Leftrightarrow \quad AX' = BX'.$$

Theorem A.74 (Albert's theorem)
Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ be symmetric. Then

(i) $A \geq 0$ *if and only if*

- (a) $A_{22} \geq 0$,
- (b) $A_{21} = A_{22} A_{22}^{-} A_{21}$,
- (c) $A_{11} \geq A_{12} A_{22}^{-} A_{21}$,

((b) and (c) are invariant of the choice of A_{22}^{-}).

(ii) $A > 0$ *if and only if*

- (a) $A_{22} > 0$,
- (b) $A_{11} > A_{12} A_{22}^{-1} A_{21}$.

Proof: Bekker and Neudecker (1989) :

(i) Assume $A \geq 0$.

(a) $A \geq 0 \quad \Rightarrow \quad x'Ax \geq 0$ for any x. Choosing $x' = (0', x_2')$ $\Rightarrow x'Ax = x_2'A_{22}x_2 \geq 0$ for any $x_2 \Rightarrow A_{22} \geq 0$.

(b) Let $B' = (0, I - A_{22}A_{22}^-) \Rightarrow$

$$\begin{aligned} B'A &= \big((I - A_{22}A_{22}^-)A_{21}, A_{22} - A_{22}A_{22}^-A_{22}\big) \\ &= \big((I - A_{22}A_{22}^-)A_{21}, 0\big) \end{aligned}$$

and $B'AB = B'A^{\frac{1}{2}}A^{\frac{1}{2}}B = 0$. Hence, by Theorem A.72 we get $B'A^{\frac{1}{2}} = 0$.

$$\begin{aligned} \Rightarrow B'A^{\frac{1}{2}}A^{\frac{1}{2}} = B'A &= 0. \\ \Rightarrow (I - A_{22}A_{22}^-)A_{21} &= 0. \end{aligned}$$

This proves (b).

(c) Let $C' = (I, -(A_{22}^-A_{21})')$. $A \geq 0 \Rightarrow$

$$\begin{aligned} 0 \leq C'AC &= A_{11} - A_{12}(A_{22}^-)'A_{21} - A_{12}A_{22}^-A_{21} \\ &\quad + A_{12}(A_{22}^-)'A_{22}A_{22}^-A_{21} \\ &= A_{11} - A_{12}A_{22}^-A_{21}\,. \end{aligned}$$

(Since A_{22} is symmetric, we have $(A_{22}^-)' = A_{22}$.)

Now assume (a), (b), and (c). Then

$$D = \begin{pmatrix} A_{11} - A_{12}A_{22}^-A_{21} & 0 \\ 0 & A_{22} \end{pmatrix} \geq 0,$$

as the submatrices are n.n.d. by (a) and (b). Hence,

$$A = \begin{pmatrix} I & A_{12}(A_{22}^-) \\ 0 & I \end{pmatrix} D \begin{pmatrix} I & 0 \\ A_{22}^-A_{21} & I \end{pmatrix} \geq 0.$$

(ii) Proof as in (i) if A_{22}^- is replaced by A_{22}^{-1}.

Theorem A.75 *If $A : n \times n$ and $B : n \times n$ are symmetric, then*

(i) $0 \leq B \leq A$ *if and only if*

(a) $A \geq 0$,
(b) $B = AA^-B$,
(c) $B \geq BA^-B$.

(ii) $0 < B < A$ *if and only if* $0 < A^{-1} < B^{-1}$.

Proof: Apply Theorem A.74 to the matrix $\begin{pmatrix} B & B \\ B & A \end{pmatrix}$.

Theorem A.76 *Let A be symmetric and $c \in \mathcal{R}(A)$. Then the following statements are equivalent:*

(i) $\operatorname{rank}(A + cc') = \operatorname{rank}(A)$.

(ii) $\mathcal{R}(A + cc') = \mathcal{R}(A)$.

(iii) $1 + c'A^- c \neq 0$.

Corollary 1: Assume (i) or (ii) or (iii) holds; then

$$(A + cc')^- = A^- - \frac{A^- cc' A^-}{1 + c'A^- c}$$

for any choice of A^-.

Corollary 2: Assume (i) or (ii) or (iii) holds; then

$$\begin{aligned} c'(A + cc')^- c &= c'A^- c - \frac{(c'A^- c)^2}{1 + c'A^- c} \\ &= 1 - \frac{1}{1 + c'A^- c}\,. \end{aligned}$$

Moreover, as $c \in \mathcal{R}(A+cc')$, the results are invariant for any special choices of the g-inverses involved.

Proof: $c \in \mathcal{R}(A) \Leftrightarrow AA^- c = c \Rightarrow$

$$\mathcal{R}(A + cc') = \mathcal{R}(AA^-(A + cc')) \subset \mathcal{R}(A).$$

Hence, (i) and (ii) become equivalent. Proof of (iii): Consider the following product of matrices:

$$\begin{pmatrix} 1 & 0 \\ c & A + cc' \end{pmatrix} \begin{pmatrix} 1 & -c \\ 0 & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -A^- c & I \end{pmatrix} = \begin{pmatrix} 1 + c'A^- c & -c \\ 0 & A \end{pmatrix}.$$

The left-hand side has the rank

$$1 + \operatorname{rank}(A + cc') = 1 + \operatorname{rank}(A)$$

(see (i) or (ii)). The right-hand side has the rank $1 + \operatorname{rank}(A)$ if and only if $1 + c'A^- c \neq 0$.

Theorem A.77 *Let $A : n \times n$ be a symmetric and nonsingular matrix and $c \notin \mathcal{R}(A)$. Then we have*

(i) $c \in \mathcal{R}(A + cc')$.

(ii) $\mathcal{R}(A) \subset \mathcal{R}(A + cc')$.

(iii) $c'(A + cc')^- c = 1$.

(iv) $A(A + cc')^- A = A$.

(v) $A(A + cc')^- c = 0$.

Proof: As A is assumed to be nonsingular, the equation $Al = 0$ has a nontrivial solution $l \neq 0$, which may be standardized as $(c'l)^{-1}l$ such that $c'l = 1$. Then we have $c = (A + cc')l \in \mathcal{R}(A + cc')$, and hence (i) is proved. Relation (ii) holds as $c \notin \mathcal{R}(A)$. Relation (i) is seen to be equivalent to

$$(A + cc')(A + cc')^- c = c.$$

Then (iii) follows:

$$\begin{aligned} c'(A + cc')^- c &= l'(A + cc')(A + cc')^- c \\ &= l'c = 1\,, \end{aligned}$$

which proves (iii). From

$$\begin{aligned} c &= (A + cc')(A + cc')^- c \\ &= A(A + cc')^- c + cc'(A + cc')^- c \\ &= A(A + cc')^- c + c\,, \end{aligned}$$

we have (v).

(iv) is a consequence of the general definition of a g-inverse and of (iii) and (iv):

$$\begin{aligned} A + cc' &= (A + cc')(A + cc')^-(A + cc') \\ &= A(A + cc')^- A \\ &\quad + cc'(A + cc')^- cc' \quad [= cc' \text{ using (iii)}] \\ &\quad + A(A + cc')^- cc' \quad [= 0 \text{ using (v)}] \\ &\quad + cc'(A + cc')^- A \quad [= 0 \text{ using (v)}]. \end{aligned}$$

Theorem A.78 *We have $A \geq 0$ if and only if*

(i) $A + cc' \geq 0$.

(ii) $(A + cc')(A + cc')^- c = c$.

(iii) $c'(A + cc')^- c \leq 1$.

Assume $A \geq 0$; then

(a) $c = 0 \Leftrightarrow c'(A + cc')^- c = 0$.

(b) $c \in \mathcal{R}(A) \Leftrightarrow c'(A + cc')^- c < 1$.

(c) $c \notin \mathcal{R}(A) \Leftrightarrow c'(A + cc')^- c = 1$.

Proof: $A \geq 0$ is equivalent to

$$0 \leq cc' \leq A + cc'.$$

Straightforward application of Theorem A.75 gives (i)–(iii).

Proof of (a): $A \geq 0 \Rightarrow A + cc' \geq 0$. Assume

$$c'(A + cc')^- c = 0\,,$$

and replace c by (ii) $\Rightarrow$

$$\begin{aligned} &c'(A+cc')^-(A+cc')(A+cc')^-c = 0 \Rightarrow \\ &(A+cc')(A+cc')^-c = 0 \end{aligned}$$

as $(A + cc') \geq 0$. Assuming $c = 0 \Rightarrow c'(A + cc')c = 0$.

Proof of (b): Assume $A \geq 0$ and $c \in \mathcal{R}(A)$, and use Theorem A.76 (Corollary 2) $\Rightarrow$

$$c'(A+cc')^-c = 1 - \frac{1}{1+c'A^-c} < 1.$$

The opposite direction of (b) is a consequence of (c).

Proof of (c): Assume $A \geq 0$ and $c \notin \mathcal{R}(A)$, and use Theorem A.77 (iii) $\Rightarrow$

$$c'(A+cc')^-c = 1.$$

The opposite direction of (c) is a consequence of (b).

Note: The proofs of Theorems A.74–A.78 are given in Bekker and Neudecker (1989).

Theorem A.79 *The linear equation $Ax = a$ has a solution if and only if*

$$a \in \mathcal{R}(A) \quad or \quad AA^-a = a$$

for any g-inverse A.

If this condition holds, then all solutions are given by

$$x = A^-a + (I - A^-A)w\,,$$

where w is an arbitrary m-vector. Further, $q'x$ has a unique value for all solutions of $Ax = a$ if and only if $q'A^-A = q'$, or $q \in \mathcal{R}(A')$.

For a proof, see Rao (1973a, p. 25).

A.13 Projectors

Consider the range space $\mathcal{R}(A)$ of the matrix $A : m \times n$ with rank r. Then there exists $\mathcal{R}(A)^\perp$, which is the orthogonal complement of $\mathcal{R}(A)$ with dimension $m - r$. Any vector $x \in \mathbb{R}^m$ has the unique decomposition

$$x = x_1 + x_2\,, \quad x_1 \in \mathcal{R}(A)\,, \quad \text{and} \quad x_2 \in \mathcal{R}(A)^\perp\,,$$

of which the component x_1 is called the orthogonal projection of x on $\mathcal{R}(A)$. The component x_1 can be computed as Px, where

$$P = A(A'A)^-A'\,,$$

which is called the projection operator on $\mathcal{R}(A)$. Note that P is unique for any choice of the g-inverse $(A'A)^-$.

Theorem A.80 *For any $P : n \times n$, the following statements are equivalent:*

(i) *P is an orthogonal projection operator.*

(ii) *P is symmetric and idempotent.*

For proofs and other details, the reader is referred to Rao (1973a) and Rao and Mitra (1971).

Theorem A.81 *Let X be a matrix of order $T \times K$ with rank $r < K$, and $U : (K - r) \times K$ be such that $\mathcal{R}(X') \cap \mathcal{R}(U') = \{0\}$. Then*

(i) $X(X'X + U'U)^{-1}U' = 0$.

(ii) $X'X(X'X + U'U)^{-1}X'X = X'X$*; that is,* $(X'X + U'U)^{-1}$ *is a g-inverse of* $X'X$.

(iii) $U'U(X'X + U'U)^{-1}U'U = U'U$*; that is,* $(X'X + U'U)^{-1}$ *is also a g-inverse of* $U'U$.

(iv) $U(X'X + U'U)^{-1}U'u = u$ *if* $u \in \mathcal{R}(U)$.

Proof: Since $X'X + U'U$ is of full rank, there exists a matrix A such that

$$\begin{aligned} & (X'X + U'U)A = U' \\ \Rightarrow \quad & X'XA = U' - U'UA \quad \Rightarrow \quad XA = 0 \text{ and } U' = U'UA \end{aligned}$$

since $\mathcal{R}(X')$ and $\mathcal{R}(U')$ are disjoint.

Proof of (i):

$$X(X'X + U'U)^{-1}U' = X(X'X + U'U)^{-1}(X'X + U'U)A = XA = 0\,.$$

Proof of (ii):

$$\begin{aligned} & X'X(X'X + U'U)^{-1}(X'X + U'U - U'U) \\ = \quad & X'X - X'X(X'X + U'U)^{-1}U'U = X'X\,. \end{aligned}$$

Result (iii) follows on the same lines as result (ii).

Proof of (iv):

$$U(X'X + U'U)^{-1}U'u = U(X'X + U'U)^{-1}U'Ua = Ua = u$$

since $u \in \mathcal{R}(U)$.

A.14 Functions of Normally Distributed Variables

Let $x' = (x_1, \cdots, x_p)$ be a p-dimensional random vector. Then x is said to have a p-dimensional normal distribution with expectation vector μ and covariance matrix $\Sigma > 0$ if the joint density is

$$f(x; \mu, \Sigma) = \{(2\pi)^p |\Sigma|\}^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(x - \mu)'\Sigma^{-1}(x - \mu)\right\}.$$